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Guide to Geometry

Maddison Webb

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Chapter 1

Euclid's Postulates

It has been shown that geometry is one of the oldest branches of math. Looking back to ancient civilizations geometry was used in a practical sense and did not have the name it has today. The Greeks were the first civilization to explore math beyond it's everyday use, simply for the sake of learning. The word geometry comes from the Greeks and translates to 'earth measure'. Pre-Euclid geometry was used to survey land, build structures, measure containers and so forth. Post-Euclid is what has shaped into the modern geometry we use today. Throughout this book constructions will be used to understand why something is true and to prove a wide range of theorems. These constructions will be completed only by using a straight edge and a compass.

Euclid's Elements

Euclid's 5 postulates are as follows:

1. A straight line can be drawn between any two points.

Figure 1.1: Postulate 1

2. Any straight line can be extended indefinitely.

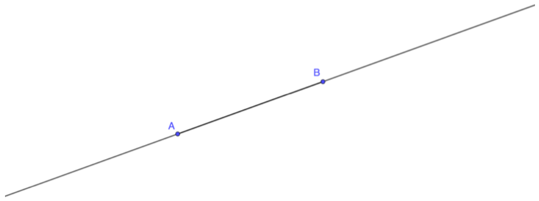


Figure 1.2: Postulate 2

3. Given any straight line segment, a circle can be drawn using the segment as the radius and one endpoint as the center.

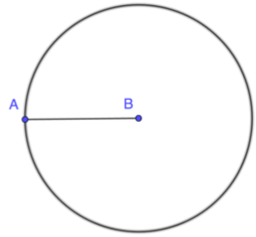


Figure 1.3: Postulate 3

4. All right angles are congruent

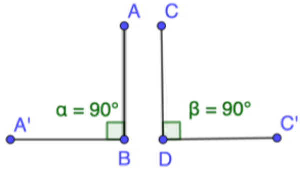


Figure 1.4: Postulate 4

5. If two lines are drawn which intersect a third line in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended indefinitely.

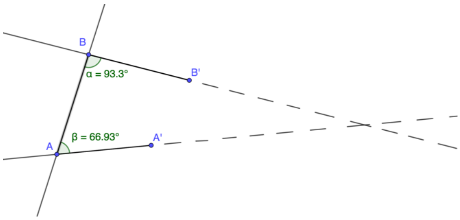


Figure 1.5: Postulate 5

These five postulates were introduced by Euclid as axioms; facts taken to be true through experimentation and logic. With the exception of the fifth postulate, they all have been proven to be true. These axioms are the basis for the Euclidean geometry used today. Compass and straight edge constructions are executed by using the first 3 of Euclid's postulates.

Chapter 2

Polygons

A **polygon** is a closed finite collection of at least 3 line segments and angles. All polygons must also have the property that each line segment intersects *exactly* two others, one at each endpoint. The most commonly thought of polygons being a triangle, square, rectangle, octagon, etc. However, 'unusual' polygons also exist, as long as they follow from the definition. Figure 2.1 shows two examples of a collection of line segments that to not form a polygon.

'A' in Figure 2.1 fails two of the properties that are required for it to be defined as a polygon. It is not closed and two of its line segments meet only one other line segment. While 'B' in figure 2.1 only fails one of these properties. It is closed, but the segments intersect more than 2 other segments.

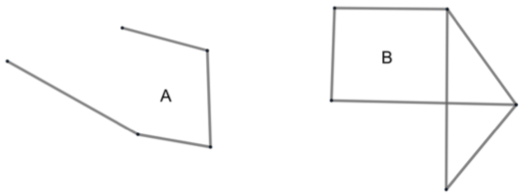


Figure 2.1: Non-examples

Both C and D from Figure 2.2 satisfy all the criteria to be classified as a polygon, despite the shape being unusual in appearance.

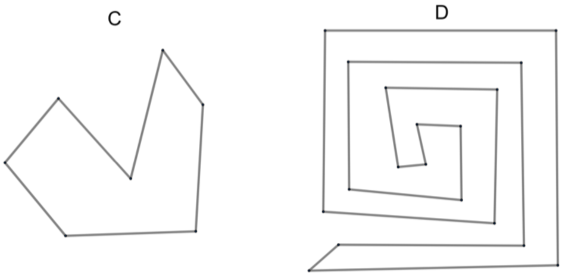


Figure 2.2: Polygon Examples

Types of Polygons

The line segments that make up a polygon are called the **sides** or **edges**, and the endpoints where the segments meet are the **vertices** of a polygon. Given any polygon it will have an n number of sides and an n number of vertices, making up what is called an n-gon. Common polygons are given names according to the number of sides and

characteristics. Some of these polygons are listed below along with it's properties.
Triangle - Any 3-gon.
Isosceles triangle - Two sides have the same length.
Equilateral triangle - All sides have the same length.
Right triangle - Contains a right angle.
Scalene triangle - Neither isosceles or right.
Quadrilateral - Any 4-gon.
Rectangle - Contains four right angles.
Square - A rectangle with four sides of equal length.
Parallelogram - Opposite sides parallel.
Rhombus - A parallelogram with four sides of equal length.
Trapezoid - Has at least one pair of parallel opposite sides.
Pentagon - Any 5-gon.
Hexagon - Any 6-gon.
Heptagon - Any 7-gon.
Octagon - Any 8-gon

This list does not include every polygon that has been given a name, but the ones that will be discussed throughout this book. A **regular polygon** is a polygon in which all sides have equal length, the term equilateral is also used in this case. A trapezoid as defined above is usually thought to look like the figure below.

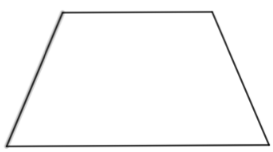


Figure 2.3: 'Traditional' Trapezoid

A trapezoid can actually vary quite a bit from Figure 2.3, a few examples are in Figure 2.4. It is defined as having *at least one* pair of parallel opposite sides. All three of the polygons below satisfy this definition, therefore they are all Trapezoids. The polygon in the middle has two pairs of parallel opposite sides satisfying the definition of a parallelogram as well.

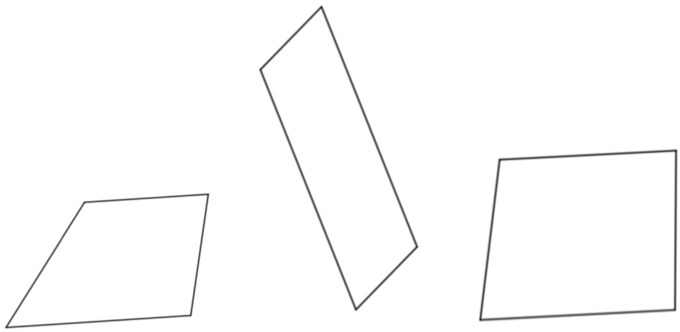


Figure 2.4: Other Trapezoids

This shows that a polygon can fit into more than one category at once.

The **perimeter** of a polygon is the total distance of the line segments that form the polygon. The perimeter of any n-gon can be found by summing the length of its sides. The **area** of a polygon is the total surface that is contained within. Finding the area of an n-gon will differ for each value of n.

Congruence

A figure X is **congruent** to figure Y (written as X ≅ Y) if they are the same shape and size. That is, one can be oriented or copied on top of the other to match up perfectly. If two figures are congruent, then so are all corresponding parts. Or, **CPCFC**: "corresponding parts of congruent figures are congruent".

Two figures are *not* congruent if any part of one figure is not congruent to the corresponding part of the other. So showing two figures are not congruent can simply be done with a counter-example.

To show two figures *are* congruent, one figure needs to have the ability to be transformed into the other . This is accomplished by using isometries. **Isometries** are movements of a figure without changing the size or shape in any way. Reflection, rotation, and translation are three types of fundamental isometries.

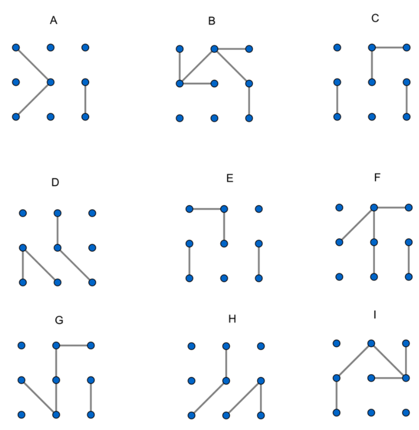
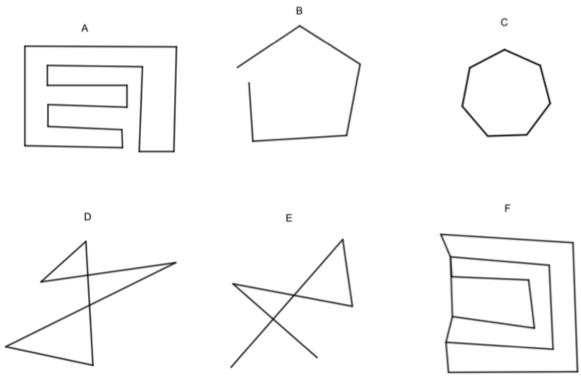


Figure 2.5:

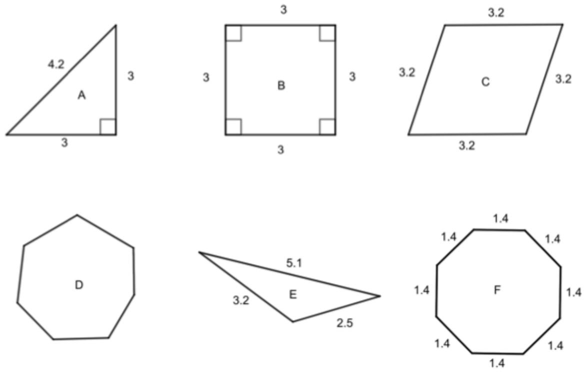
If we take 'B' from 2.5 and reflect it we can see that it is very similar to 'I'. But, because of that one extra line segment, they are not congruent. The definition states they have to line up *exactly*. If only one part is not congruent to another then the figures do not have congruence property. Where if D is reflected it will line up perfectly with H, CPCTC so $D \cong H$.

Exercises

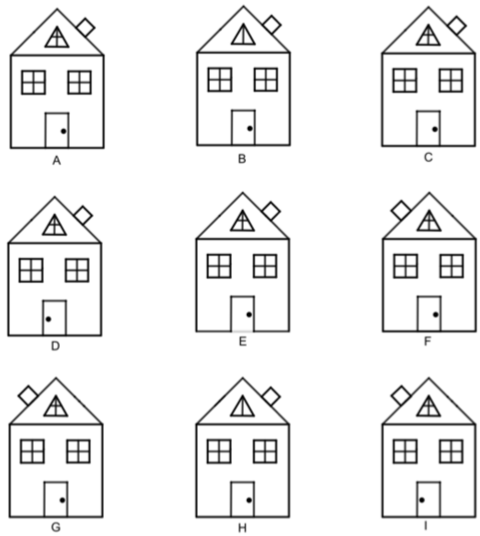
1. For each figure below determine if it satisfies all the properties of a polygon. If not, which properties are not satisfied?



2. Identify each of the polygons below. (Recall a polygon can have more than one identification.)



3. Which of the following figures are congruent?



4. What combination of isometries will show the congruence of the two figures?

Chapter 3

Fundamentals of Euclidean Geometry

As mentioned in Chapter 1, constructions will be used to better understand the 'why'. When asked to construct:

- Only a pencil, straight edge, and compass may be used.
- Use postulates 1-3 find an approach to complete the problem given.
- Organize a list of the steps that are completed in a way the reader can clearly see the process.
- Prove why your method is valid in producing the desired result. Only use what is known to be true through construction itself. That is, no conjectures can be made. Note: A theorem that has been previously proven can be used in the logic of a proof.

Distance

The **distance** from point A to point B is equal to the length of the line segment AB. Distance, denoted by $\mathcal{D}(AB)$, has the following properties:

- $$\mathcal{D}(AB) \geq 0 \tag{3.1}$$

- $$\mathcal{D}(AB) = \mathcal{D}(BA) \tag{3.2}$$

- $$\text{If } \mathcal{D}(AB) = 0, \text{then } A = B \tag{3.3}$$

Circles

A circle is *not* a polygon. A circle can be formally defined using postulate 3. Let AB be some segment with a length of r. A **circle** is the set of all points whose distance from B is r. Point B is the **center** of this circle, while r is the **radius**. This circle is illustrated in Figure 3.1,

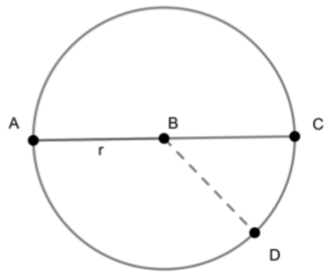


Figure 3.1: Caption

Some properties of circles can follow directly from the definition. If two circles have the same radius they are congruent. All segments that are formed with the center and *any* point lying on the circle are congruent. So in the figure above, $AB \cong BC \cong BD$. Given that the center, B, is between A and C with A and C lying on the circle, segment AC is **a diameter** of the circle. **The diameter** of this circle is the length of AC. This is also shown in Figure 3.1. Using definitions, the diameter of a circle can be generalized:

$$\begin{aligned}
\mathcal{L}(AB) &= r. \\
\text{since } AB &\cong BC, \text{ then} \\
\mathcal{L}(AB) &= \mathcal{L}(BC) \\
\mathcal{L}(AC) &= \mathcal{L}(AB) + \mathcal{L}(BC) \\
&= r + r(\text{ by substitution}) \\
&= 2r
\end{aligned}$$

$\therefore$ Diameter is twice the radius.

Angles

An **angle** is a pair of rays that share the same endpoint. Every angle is assigned a positive number between 0° and 180°, this is the **angle measure** (Denoted $\mathcal{M}(\angle A)$). Angles are congruent only if they have the same measure.

Angles can be classified into four types of angles:

A **straight** angle measures exactly 180°.

An **acute** angle measures less than 90°.

An **obtuse** angle measures greater than 90°.

A **right** angle measures exactly 90°.

If D is between A and C then

$$\mathcal{M}(\angle ABC) = \mathcal{M}(\angle ABD) + \mathcal{M}(\angle DBC)$$

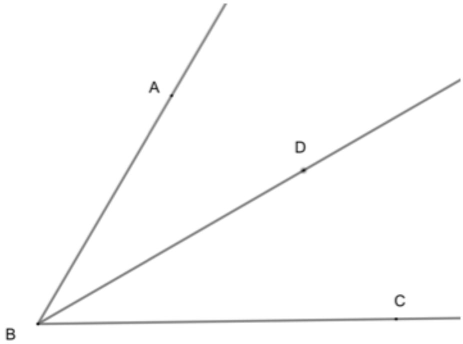


Figure 3.2: Caption

The ray BD is the **bisector** of $\angle ABC$ if D is in the interior of $\angle ABC$ and $\angle ABD \cong \angle DBC$. Two angles are considered **complimentary** if they sum to 90°, if their sum is 180° they are defined as **supplementary** angles. Figure 3.3 gives an example of **vertical** angles. E is between A and B as well as between C and D creating vertical angles. Vertical angles are congruent, so in the figure below $\alpha \cong \beta$.

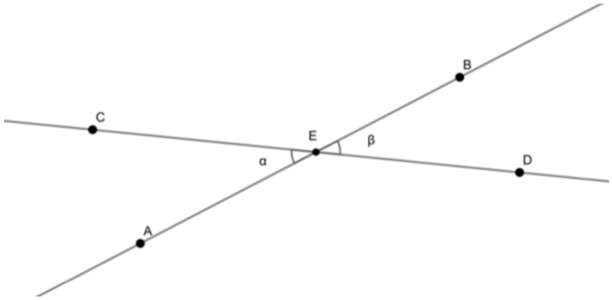


Figure 3.3: Vertical Angles

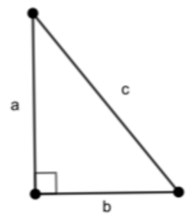
Pythagorean Theorem

For this section, the area of a square and a triangle need to be well defined.

$$\text{Area of a square} = base \times height$$
$$\text{Area of a triangle} = \frac{1}{2} \times base \times height$$

Theorem. *Let a, b, and c be any positive real number. If a right triangle, T, has sides of length a and b, with hypotenuse c, then*

$$a^2 + b^2 = c^2 \tag{3.4}$$



This theorem has been proved with many different methods. Continuing with proof by construction, a square of side length a + b (Figure 3.4) will need to be constructed.

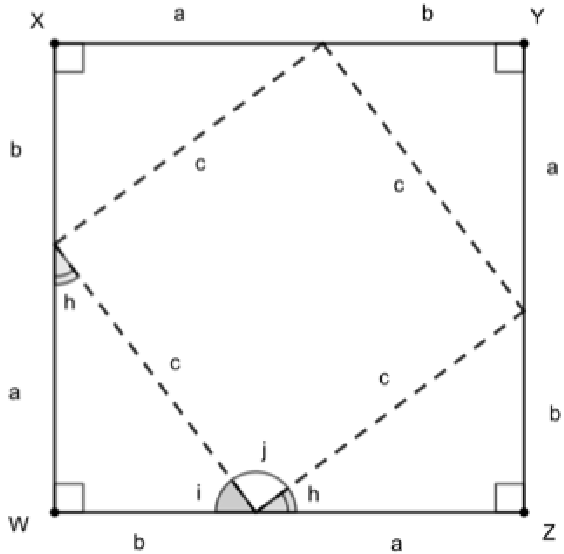


Figure 3.4: Square with Side Length a + b

Proof:

It needs to be shown that in Figure 3.4, the smaller dotted quadrilateral and the larger quadrilateral are squares. Each side can be represented by a + b. Each a and b were constructed in a way that

$$a \cong a \text{ and } b \cong b \implies a + b = a + b$$

$$\therefore WX \cong XY \cong YZ \cong ZW$$

By construction, the interior angles of the quadrilateral are right angles $\therefore$ Figure 3.4 is a square.

Triangle T is a right triangle so $\angle XWZ$ is 90°

$$\angle XWZ + \angle h + \angle i = 180^\circ \text{ (Definition of a Triangle)}$$

$$90 + \angle h + \angle i = 180^\circ \text{ (Substitution)}$$

$$\angle h + \angle i = 90^\circ$$

Angles h, i, and j make up a straight line

$$\begin{aligned} \angle h + \angle i + \angle j &= 180^\circ \text{ (Definition of supplementary angles)} \\ 90 + \angle j &= 180^\circ \text{ (Substitution)} \\ \therefore \angle j &= 90^\circ \end{aligned}$$

By similar argument, the remaining interior angles of the dotted equilateral are right angles. By definition it is a square as well.
The area of the large square can be computed in two ways. First by the definition of a square's area:

$$\begin{aligned} A &= (a+b)(a+b) \\ &= a^2 + 2ab + b^2 \end{aligned}$$

The other way to calculate the total area is by summing the area of the triangle and smaller square.

$$\begin{aligned} A &= 4(\frac{1}{2}ab) + cc \\ &= 2ab + c^2 \end{aligned}$$

Setting the areas equal to each other:

$$\begin{aligned} a^2 + 2ab + b^2 &= 2ab + c^2 \\ a^2 + b^2 &= c^2 \end{aligned}$$

□

Parallel Lines

Euclid's fifth postulate has been written in many ways throughout the years in an attempt to prove it. One of these gives the definition of parallel lines. Giving this postulate a second name: 'Euclid's parallel postulate'. **Parallel** lines are lines that do not intersect at any point when extended indefinitely. Given two parallel lines, when those lines are intersected by a transversal line relationships with the angles arise.

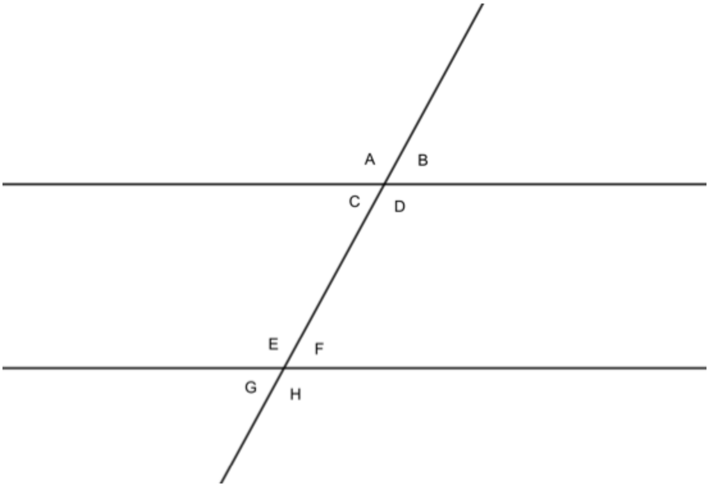


Figure 3.5: Caption

Using Figure 3.5 as an example, here are the different types of angle pairs given by a transversal:
Corresponding angles - B and F, A and E, D and H
Alternate interior angles - C and F, D and E
Same side interior angles - C and E, F and D
Alternate exterior - A and H, B and G
Two lines are parallel if and only if when cut by a transversal the following are true:
All pairs of corresponding angles are congruent.
All pairs of alternate interior angles are congruent.
All pairs of alternate exterior angles are congruent.
All pairs of same side interior angles are supplementary.

Interior Angles of Polygons

As discussed in the previous chapter, the sides or edges of a polygon are made up of line segments. An **interior angle of a polygon** is an angle inside the polygon made up of two adjacent line segments. An n-gon has n interior angles just as it has n sides and n vertices.

Theorem. *The interior angles of a triangle sum to 180 °.*

The proof of this theorem is left as an exercise.
Now thinking about quadrilaterals, the definition of both a rectangle and a square states that it has four right angles. It is easily seen that a rectangle's interior angles sum to 360 ° ($90(4) = 360$). What about the other types of quadrilaterals? Take any two quadrilaterals:

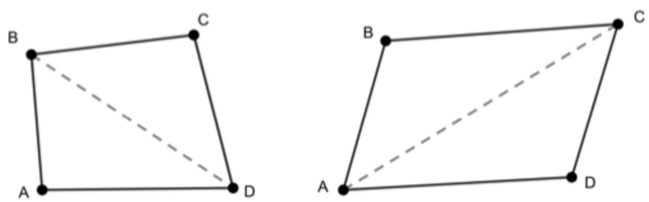


Figure 3.6: Caption

A segment can be formed from B to D breaking this quadrilateral into two triangles. This segment is called a **diagonal**. A diagonal is a line segment connecting two non-adjacent vertices. A diagonal also exists from A to C, again splitting the polygon into two triangles. Using the theorem about interior angles of a triangle, it can be shown that, for any quadrilateral, the interior angles sum to 360 °. This process of splitting a polygon into triangles is known as **triangulation**.

Congruence of Triangles

In order for polygons to be congruent they need to follow the same general equation, CPCTC. Congruence of circles is quite easy, if they have the same radius they are congruent. Triangles on the other hand are more complicated. Given the two triangles below, they are congruent only if $AB \cong DE, BC \cong EF, AC \cong DF, \angle BAC \cong \angle EDF, \angle ABC \cong \angle DEF, \angle ACB \cong DFE$. Congruence is represented with the symbols shown in Figure 3.7, so $\triangle ABC \cong \triangle DEF$.

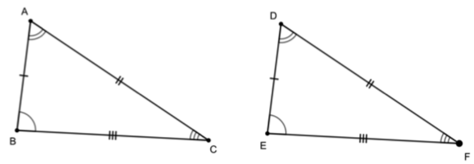


Figure 3.7: Congruent Triangles

Theorem. ***Side-Side-Side (SSS)****If the three sides of one triangle are congruent to the corresponding three sides of the other triangle, then the triangles themselves are congruent.*

Theorem. ***Side-Angle-Side (SAS)****If two sides and the angle between them of one triangle is congruent to the corresponding parts of the other triangle, then the trianges themselves are congruent.*

Theorem. ***Angle-Side-Angle (ASA)****If two angles and the side between them of one triangle is congruent to the corresponding parts of the other triangle, then the triangles themselves are congruent.*

Theorem. ***Angle-Angle-Side (AAS)****If two angles and a side not between them of one triangle is congruent to the corresponding parts of the other triangle, then the triangles themselves are congruent.*

Exercises

1. Verify Euclidean distance satisfies the properties of distance.

2. Construct an angle bisector.

3. Construct an equilateral triangle.

4. Does Angle-Angle-Angle guarantee congruence? What about Side-Side-Angle? Why or why not?

5.

Chapter 4

Similar Figures

Two figures are **similar** if one is congruent to the other after a dilation of a scaling factor, k. They have the same shape but are not the same size. '$\sim$' is used to represent this relationship (A $\sim$ B). Congruent figures are also similar, k = 1. All circles will be similar to any other circle since a k can always be chosen to dilate, making the radii congruent.

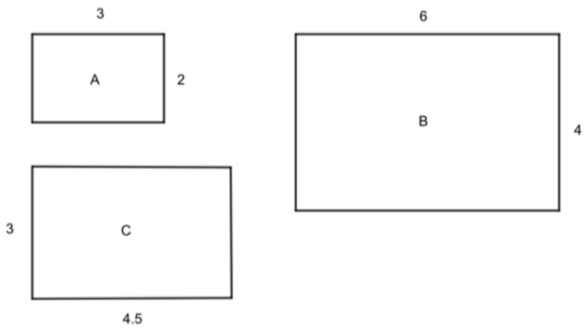


Figure 4.1: Similar Rectangles

Three similar rectangles are shown in Figure 4.1. If A is dilated with a k of 2 (doubling the size) B is the result so A$\sim$ B. Dilating A by $\frac{1}{2}$ makes it congruent to C, therfore A $\sim$ C. In this figure A $\sim$ B $\sim$ C.

Theorem. ***Similar Triangles*** *if* $\triangle ABC \sim \triangle DEF$*, then*

$$\frac{AB}{AC} = \frac{DE}{DF}$$

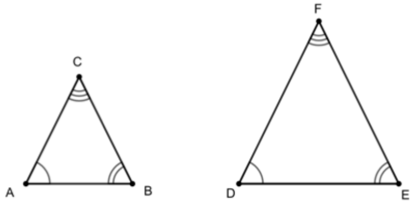


Figure 4.2: Similar Triangles Theorem

Theorem. ***Side-Angle-Side Similarity*** *If* $\triangle ABC$ *and* $\triangle DEF$ *are two triangles where* $\angle CAB \cong \angle FDE$ *and* $\frac{AB}{AC} = \frac{DE}{DF}$*, then* $\triangle ABC \sim \triangle DEF$*.*

Theorem. ***Angle-Angle-Angle Similarity*** *If corresponding angles of two triangles are congruent, then the triangles are similar.*

If a parallel line intersects a triangle, it creates similar triangles as well. Line m is parallel to line l and creates a second triangle, $\triangle ADE$. By definition of a transversal through parallel lines, cooresponding angles are congruent. $\angle AED \cong \angle ABC$ and $\angle ADE \cong \angle ACB$, this shows $\triangle ABC \sim \triangle AED$.

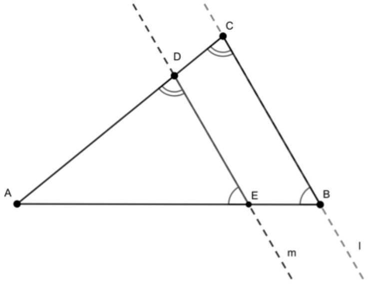


Figure 4.3: Triangle Cut by a Parallel Line

Chapter 5

Trigonometry

Trigonometry is formed by the fundamentals of similar triangles and the pythagorean theorem. It deals with the relations of the sides and angles of triangles and the relevant functions of any angles.

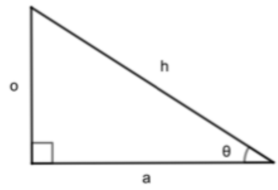


Figure 5.1: Right Triangle with $\theta°$ angle

In the right triangle above, o, h, and a stand for opposite, hypotenuse, and adjacent respectively. These are oriented based off the angle θ.The following ratios are true for such a triangle.

$$sin(\theta°) = \frac{o}{h}, \qquad cos(\theta°) = \frac{a}{h}, \qquad tan(\theta°) = \frac{o}{a} \tag{5.1}$$

Theorem. ***(Pythagorean Identity)*** *For any angle θ:*

$$sin^2(\theta) + cos^2(\theta) = 1 \tag{5.2}$$

Both the ratios and Pythagorean identity are only true for right triangles, where the following theorems are true for any triangle.

Theorem. ***(Law of Sines)*** *If $\triangle ABC$ is any triangle, then*

$$\frac{sin(A)}{a} = \frac{sin(B)}{b} = \frac{sin(C)}{c} \tag{5.3}$$

Theorem. ***(Law of Cosines)*** *If $\triangle ABC$ is any triangle, then*

$$c^2 = a^2 + b^2 - 2abcos(C) \tag{5.4}$$

Chapter 6

Tessellations

A **tessellation** is a pattern made up of various shapes, completely covering a plane with no gaps or overlaps. A single shape **tessellates the plane** if copies of itself can be pieced together to cover the plane with no gaps and no overlaps. For example, a circle does not tessellate the plane, there is no way it can be copied to cover the whole plane as shown in Figure 6.1. Where in Figure 6.2 an equilateral triangle is show tessellating a plane. There is no overlap and no gaps exist.

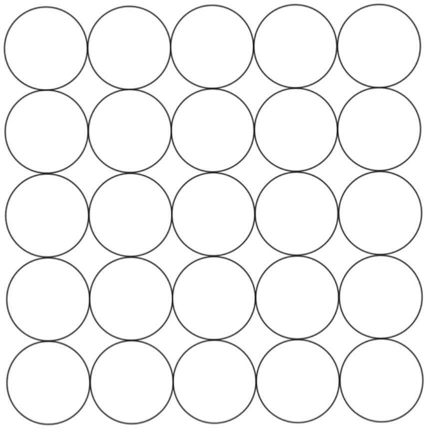


Figure 6.1: Non-example

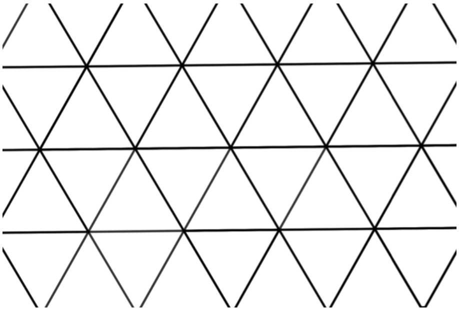


Figure 6.2: Example

Chapter 7

Non-Euclidean Geometry

Non-Euclidean geometry can be broken down into two main branches: spherical geometry and hyperbolic geometry. While Euclidean geometry is based on a plane without curvature, both spherical geometry and hyperbolic geometry have curvature. Spherical geometry has positive curvature, meaning that it curves the same way in every direction regardless of starting point. Hyperbolic geometry, on the other hand, has negative curvature; the curvature will be different depending on the direction.The basis for Non-Euclidean geometry uses the first four of Euclid's postulates while replacing the fifth with the negation.

When mathematicians of the 19th century explored the negation, the results came to be very surprising. For example, 'straight' lines are now curved, similar triangles are now congruent, and the angles of a triangle do not sum to be 180 degrees. These results, which were so surprising to the mathematicians of this time, developed to be the fundamentals of Non-Euclidean geometry.

Spherical Geometry

The general form of of a sphere is

$$x^2 + y^2 + z^2 = k \tag{7.1}$$

In this equation, k is the radius. The larger k becomes, the larger the sphere will become. If k is equal to zero then the sphere is a single point, if k is less than zero the sphere does not exist.

A 'straight' line is defined as the shortest distance from one point to another. In Euclidean geometry, this distance will always be a line that is 'straight' to an individual's eye, but in Non-Euclidean geometry this distance will have a curvature to it. In spherical geometry, using the intersection of a plane through the center of the sphere will generate a straight line; this intersection yields a circle. This circle intersected on this plane is defined as a **great circle**; all straight lines in spherical geometry are great circles.

Hyperbolic Geometry

Hyperbolic geometry can be quite hard to visualize. There are a couple of famous models that have been projected from a three-dimensional model. These projections let us look at hyperbolic geometry in a Euclidean sense, making it easier to visualize. The general equation of a sphere (7.1) modified slightly becomes the general equation for Hyperbolic.

$$x^2 + y^2 - z^2 = k \tag{7.2}$$

In this equation k acts quite differently than in the spherical equation. When k is equal to zero, a cone is formed through the origin. K equal to one produces a single-sheet hyperboloid. K equaling negative one will yield a two-sheet hyperboloid, two separate components, one positive and one negative. The individual components of the two-sheet hyperboloid are referred to as hyperbolas. The positive hyperboloid is the main focus for the hyperbolic geometry model. These three k values construct the **Minkowski Model**.

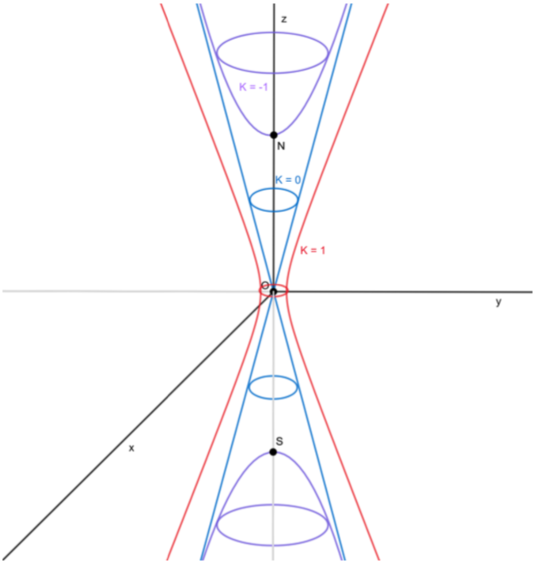


Figure 7.1: Minkowski Model

A projection of the Minkowski model from the view of the plane through the origin results in a nice Euclidean model. From this view, the model is projected onto a plane through the point N.

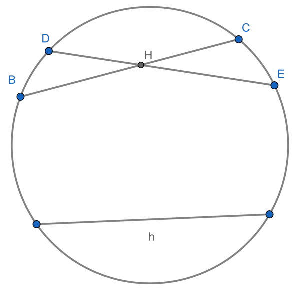


Figure 7.2: Projection on plane through N

In this projection, the disk projected is bounded by the cone, or K=0. Every point in the hyperbola will project directly onto the new plane. The parabolas from the hyperbola will project as straight lines. This model preserves straight lines, but distorts angles.

This model from 7.2 consists of a disk which exists in the Euclidean plane. All points in this model exist within the disk and all lines are conveyed as chords. A chord is defined as a straight line with the endpoints on the boundary of the disk. Parallel lines in the model are defined as lines that fail to intersect within the disk. For example, in 7.2 the line ‘L’ and the line ‘AB’ are considered to be parallel. A line being considered perpendicular to another is dependent on if one of these is a diameter. If one of these lines is a diameter, then the perpendicularity is that of the Euclidean sense. If neither is a diameter, then they are only perpendicular if the Euclidean extension of one passes through the **pole** of the other. The **pole** of a line is defined as the point where the endpoint of the line and the tangents of the disk intersect.

Another projection of the Minkowski model is from the view of the plane intersecting point S in . From this view, the model can be projected onto the x-y plane through the origin.

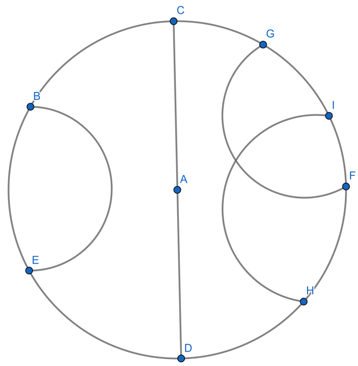


Figure 7.3: Projection on plane through the Origin

This will create a disk bound by the single-sheet hyperboloid in the three-dimensional model, or K = 1. The straight lines drawn through the positive hyperbola will project as arcs; these arcs will be perpendicular to the boundary. Even though the model might appear to be finite, the disk bounding the model is the infinite hyperboloid K = -1 as shown in . So, even though it looks like a point has reached the boundary; it can only approach the edge of the disk and will never actually lie directly on it. This model does not preserve 'straightness' therefore the angles remain true.

The model in 7.3 is an n-dimensional model of hyperbolic space. The points are on an n-dimensional disk with two types of lines that can occur. One of these lines is simply a diameter of the disk. Just like the previous model in 7.2, parallel lines in this new model are defined as arcs that never meet or intersect. In order for the two lines to be perpendicular in model 7.3 they must meet orthogonally. If two arcs intersect on the boundary of the disk they are called limit rays. When three arcs intersect in the disk it creates a hyperbolic triangle. Unlike in Euclidean geometry, hyperbolic triangle's angles do not sum to 180 degrees. The angle sum of a hyperbolic triangle does not yield a consistent sum; but instead will sum to any number less than 180. This fact causes similar hyperbolic triangles to be congruent. This chapter is only an introduction to what happens if Euclid's 5th postulate is taken to be false. It is suggested to explore this idea using Non-Euclid software.