4-1966

INTUITIONIST LOGIC

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A Thesis Submitted to the Graduate Division in Partial Fulfillment of the Requirements for the Degree of Master of Arts

By

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Pittsburg, Kansas

April, 1966

CHAPTER I

INTRODUCTION

This thesis deals with the somewhat controversial topic of intuitionist logic and mathematics. Since the writer's objective in this case is to explain this theory, the point of view expressed herein will tend toward that of the intuitionist's, with emphasis given to the positive aspects of the theory.

For those who might wonder whether the material covered in this thesis is really logic, mathematical philosophy, or foundations of mathematics; there is no exact answer. These three concepts tend to be artificial since the boundries between them, if any, are rather vague. The writer does feel though, that this material should be pertinent to any understanding of that which is called "mathematics" (whatever that may mean).

Due to the nature of the material being considered, the writer feels that a survey of the historical as well as the philosophical background to the problem will give the reader somewhat of a perspective from which to examine the intuitionist theory. Thus, much of the material presented will be of a more general nature with the areas of specific concentration being mainly the logic from intuitionism and its comparison to a classical mathematical logic. With this short introduction, the historical problem will be presented.

Let those who come after me wonder why
I built up these mental constructions and
how they can be interpreted in some philosophy;
I am content to build them in the conviction
that in some way they will contribute to
the clarification of human thought.

--Arend Heyting

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CHAPTER II

HISTORICAL BACKGROUND

The nineteenth century could, for the most part, be considered the period in which pure mathematics was born. Even though its antecedents reach far back into history, it was never really distinguishable from applied mathematics and science until this time. One of the great contributions in the development of pure mathematics was that of Boole, who in 1854 developed the idea of a symbolic or formal logic. This resulted in more abstraction and intensified formalization of mathematics. Using this, as well as other logico-mathematical methods, mathematicians began to attack the foundations of the various fields of mathematics in an attempt to eliminate inconsistencies and secure an absolute mathematical system.

One related problem at this time was the perplexing question concerning the consistency of the newly developed non-Euclidean geometries. In attempting to determine the consistency of these new geometries, it was shown by certain mathematicians that the proof of consistency of these systems could be reduced to that of proving the consistency of Euclidean geometry. In other words, these non-Euclidean geometries are consistent if Euclidean geometry is consistent. This correspondence of consistencies was accomplished by devising a "model" (or interpretation) for the postulates so that each postulate is shown to be a statement holding true

for the model. For example, the consistency of plane Riemannian geometry can be established by using as a model the surface of a Euclidean sphere. Each Riemannian postulate is thus converted into a truth of Euclid. Now the question of Riemannian consistency is reduced to that of Euclidean consistency. This procedure, used for establishing consistency, is a powerful method. But it is still vulnerable since it is only a relative proof which depends upon the consistency of another system.

From advances in the field of analytic geometry, it was shown that a strong correspondence exists between Euclidean geometry and algebra by use of the Cartesian coordinate system. Then, late in the nineteenth century, David Hilbert was able to establish Euclidean geometry in terms of algebraic "truths." From this, the question of Euclidean consistency was then reduced to that of algebraic consistency.

From further developments at this time, it was shown that algebra as well as other fields of mathematics could be developed from the more fundamental theory of arithmetic. Such was the important breakthrough of Cantor by his development of set theory and the theory of transfinite numbers, which enabled mathematicians to base the foundations of analysis and calculus on that of arithmetic. From this, analysis and calculus could be developed from the basic concepts and operations used in arithmetic. At this time many mathematicians had the exhilarating conviction that it should be possible for one single axiomatic system to be developed which would yield all the traditional branches of mathematics.

Mathematicians now began to center their attention on the foundations of elementary arithmetic. The mathematician Peano was the first to reduce the theory of arithmetic to the smallest number of postulates and undefined terms. His system consisted of five basic postulates and the three undefined terms "zero," "number," and "successor." Using only these propositions and symbolic logic, he was able to define the natural numbers 0, 1, 2, 3, . . . from which he could develop elementary arithmetic. The question of consistency was thus reduced to these basic assumptions.

Around this time the mathematician Frege attempted to reduce the theory of natural numbers even further; to that of logic and set theory. It was Frege's belief that all of mathematics could be developed from logic (which included Cantor's set theory). Thus Frege worked for over twelve years to develop his thesis in a two-volume treatise on the foundations of mathematics. But even while his work was going to press, he received a letter from Bertrand Russell which shattered his dream, for Russell had found an antinomy or paradox in the set theory which Frege had employed.

This antinomy, known as "Russell's paradox," follows in this way: it seems that sets can be of two kinds, those which contain themselves as elements, and those which do not. A set will be called "normal" if, and only if, it does not contain itself as a member. If it does contain itself as a member, it will be called "non-normal." As an example, the

set of all men is a normal set, since the set itself is not a man and is therefore not a member of itself. An example of a non-normal set is the "set of all sets," since by definition it is a set and therefore must be contained as a member of itself. Let N be defined as the set of all normal sets. The question now arises as to whether N itself is a normal set. If it is a normal set, by definition of N it must be an element of itself; but, in this case, N is non-normal since any set which contains itself as a member is non-normal. On the other hand, if N is non-normal, it is a member of itself by definition of being non-normal; but, in this case, N is normal since N contains only normal sets. By either possible hypothesis a contradiction is reached. Soon afterward other such antinomies were discovered.

At this time there were three main schools of mathematical thought, each with its own philosophical interpretation of the nature of mathematics. With the discovery of the antinomies in set theory, each school thus developed a logico-mathematical theory which attempted to restore consistency to mathematics. These three schools of thought, known as logicism, formalism, and intuitionism will be briefly examined.

CHAPTER III

LOGICISM

The tendency of thought known as logicism follows the doctrine that all of the theory of mathematics is derivable from, or can be reduced to, logic alone. Frege and Russell were the main formulators of this view. Rather than just being a tool of mathematics, logic becomes the foundations and progenitor of mathematics. Mathematical concepts are formulated in terms of logical concepts, and the theorems of mathematics are developed as theorems of logic.

The logistic thesis arises from the effort to push down the foundations of mathematics to as basic a level as possible. Using Peano's theory, Frege had attempted to reduce these axioms to axioms of logic in his two-volume work. With the occurrence of the set antinomies, a more detailed and rigorous treatment was needed. This was supplied by the monumental _Principia Mathematica_ of Whitehead and Russell (1910-1913). This great work purports to provide a detailed reduction of the whole of mathematics to logic.

The _Principia Mathematica_ (hereafter denoted P.M.) begins its formal abstract development with certain postulates and undefined terms which it calls "primitive propositions" and "primitive ideas." These primitive propositions and ideas are not to be subjected to interpretation, but are to be regarded (for the most part), as hypotheses related to the real

world. Thus, no attempt is made to prove the consistency of the primitive propositions since they follow from a concrete rather than an abstract point of view.

From these primitive propositions and ideas, using a calculus of propositions, P.M. attempts to present a system which proceeds up through the theory of sets and relations to the establishment of natural numbers, from which the rest of mathematics can be derived. By this development, it is stated (by Russell) that the natural numbers emerge with the unique meanings which are ordinarily associated with them, rather than being defined nonuniquely as any things which satisfy a certain set of abstract postulates.

To avoid the paradoxes of set theory, P.M. employs a "theory of types." As a rough indication of what is meant by a "type," it may be said that individuals, sets of individuals, relations between individuals, relations between sets, sets of sets, etc., are of different types. Using this theory of types, such statements as "sets being or not being members of themselves" are rendered meaningless by the requirement that sets must contain members of only one type.

To digress slightly at this point, the actual logic of P.M. will be examined to provide a more concrete idea of the nature of mathematical logic. Also, to be able to better understand the logic developed by the intuitionists, an examination of this logic will provide a background which can be used to contrast with the intuitionist theory. The logic from

P.M. is chosen since it is basically that which is employed throughout classical mathematics.

CHAPTER IV

LOGIC FROM PRINCIPIA MATHEMATICA

In what follows, a sketch will be given of the basic program carried out in P.M. Only enough detail will be given to illustrated the general procedure followed, as well as to furnish a basis for the later discussion of intuitionist logic. Thus, only the most pertinent ideas and concepts will be presented.

In dealing with the axiomatic method, one starts with certain terms left undefined, and with certain basic formal assumptions. However, in P.M., Russell and Whitehead provide an auxiliary explanation of the "meanings" of these primitive ideas and propositions. This is necessary, since they are not dealing with such matters where one can usually assume that the reader is already familiar with the basic ideas as in, say, Euclidean geometry. These explanations do not constitute definitions since they involve the ideas they explain.

The most elementary notion used in P.M. is that of "proposisions." A proposition is a statement involving only definite notions with no variables. Thus, "this pen is green" is a proposition. The letters p, q, r, s are used to denote elementary propositions. Statements that contain variables and which become propositions when specific constants are substituted for all variables, are called propositional functions.

Propositions are assigned "truth values"; truth if it is true, and falsehood if it is false.

Another notion is that of the assertion of a proposition or propositional function. This is indicated by the symbol "$\vdash$". Thus,

$$\vdash . p$$

asserts the truth of the proposition p. All postulates are assertions, and therefore are preceded by the symbol $\vdash$. Any proposition stated in symbols without the assertion sign "$\vdash$" is merely put forward for consideration and is not asserted.

Dots on line with the symbols have two uses: one is to serve as parentheses and brackets; e.g., ".pvq." instead of "(pvq)". For brackets enclosing parentheses, the symbol ":" is used. Thus, when ":" occurs in a formula, it indicates a bracket enclosing everything to the next ":" or to the end of the expression. The other use of a dot is to indicate the logical product of two propositions; e.g. "p.q", which has the meaning "p and q", i.e., "p is true and q is true."

Negation, which uses the symbol "$\sim$", is explained as follows: if p is any proposition, then "$\sim$p" represents the proposition "not-p" or "p is false." Thus, if p is true, $\sim$p is false; and if p is false, $\sim$p is true. In this paper, the representation "p'" will be used for "not-p". This will be the only symbol of major use differing from P.M.

Disjunction is as follows: If p and q are two propositions, the proposition "p or q", i.e., "either p is true

or q is true," where the "or" is the inclusive, "and/or" will be represented by

$$p \vee q$$

This is called the "disjunction" or "logical sum" of p and q. Similarly, "p'vq" will mean "p is false or q is true."

P.M. employs three basic formal definitions. Symbolically they employ the sign "=" along with the letters "R.Df" which together indicate a definition.*

The three definitions are as follows:

R.Df.1	$p \supset q . = . p' \vee q$
R.Df.2	$p . q . = . (p' \vee q')'$
R.Df.3	$p \equiv q . = . p \supset q . q \supset p$

R.Df.1 is the definition of implication, which uses "$\supset$" as its basic symbol. It is stated that p implies q when the proposition q follows from a proposition p, so that if p is true, q must be true. This property does not determine what is implied by a false proposition though. What it does determine is that if p implies q, then it is impossible for p to be true and q false, i.e., it must follow that either p is false or q is true. Otherwise, the system would be inconsistent. Thus, the definition is interpreted to mean: "Either p is false or q is true." This type of implication is sometimes called "material implication," since the truth-value of $p \supset q$ is determined solely by the constituent propositions p and q, and there

*The "R" will be used to emphasize that the definition is from P.M. The R will also be used in numbering all axioms and theorems taken from P.M.

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added and the resulting proposition is true.

R.Pp.3: If either p or q is true, then either q or p is true. In P.M., this is called the "principle of permutation." This is essentially the commutativity of v.

R.Pp.4: If p is true, or either q or r is true, then q is true, or either p or r is true. This is a special type of associative law.

R.Pp.5: If q implies r, then (p or q) implies (p or r). This is called the "principle of summation" in P.M.

These primitive propositions, from the logistic viewpoint, are basic logical "truths," and are to be considered true regardless of the truth values of the propositions p, q, and r. The name given for such expressions is "tautology." With these tautologies, however, certain rules are needed for determining their own consequences.

There are two basic rules by means of which theorems can be derived from given tautologies. These rules will be denoted as follows:

Sub.: (rule of substitution). Substitution of a proposition or a formula for all occurrences of a proposition in a given formula. (For example, substitution of pvq in R.Pp.2 for q gives $(p \vee q) . \supset . p \vee (p \vee q)$). Also, substitution of any expression which is identical to a given expression by definition.*

m.p.: (modus ponens or rule of inference). If given

*This rule is not given as a general rule in P.M., but is employed throughout for particular cases.

" $\vdash .p$" and " $\vdash .p.\supset.q$", then this implies " $\vdash .q$".

With these rules and primitive propositions, some theorems will now be derived. The first will be the law of the excluded middle which will be derived through a sequence of lemmas.

R.1	$\vdash :.q\supset r.\supset:p\supset q.\supset.p\supset r$	
Proof:	$\vdash :.q\supset r.\supset:p\vee q.\supset.p\vee r$	(R.Pp.5)
	$\vdash :.q\supset r.\supset:p'\vee q.\supset.p'\vee r$	(sub. p' for p)
	$\vdash :.q\supset r.\supset:p\supset q.\supset.p\supset r$	(R.Df.1)

This theorem is one form of the transitivity of implication, sometimes called the "principle of the syllogism." Thus, if propositions of the form $X\supset Y$ and $Y\supset Z$ are asserted, and $X\supset Z$ is to be proven, then the proof would be as follows: (call this proof Syll.)

Syll. Proof:	$\vdash :.q\supset r.\supset:p\supset q.\supset.p\supset r$	(R.1)
	$\vdash :.Y\supset Z.\supset:X\supset Y.\supset.X\supset Z$	(sub. X, Y, Z for p, q, r respectively)
	$\vdash .Y\supset Z$	(given)
	$\vdash .X\supset Z$	(m.p.)
R.2	$\vdash :p.\supset.p\vee p$	
Proof:	$\vdash :q.\supset.p\vee q$	(R.Pp.2)
	$\vdash :p.\supset.p\vee p$	(sub. p for q)

R.3 $\vdash .p \supset p$

Proof: $\vdash :. q \supset r . \supset : p \supset q . \supset . p \supset r$ (R.1)

$\vdash :: p \vee p . \supset . p : \supset :. p . \supset . p \vee p : \supset . p \supset p$ (sub. pvp for q and p for r)

$\vdash . p \vee p . \supset p$ (R.Pp.1)

$\vdash :. p . \supset . p \vee p : \supset . p \supset p$ (m.p.)

$\vdash : p . \supset . p \vee p$ (R.2)

$\vdash . p \supset p$ (m.p.)

R.4 $\vdash . p' \vee p$

Proof: $\vdash . p \supset p$ (R.3)

$\vdash . p' \vee p$ (R.Df.1)

R.5 $\vdash . p \vee p'$ Law of the excluded middle.

Proof: $\vdash : p \vee q . \supset . q \vee p$ (R.Pp.3)

$\vdash : p' \vee p . \supset . p \vee p'$ (sub. p' for p and p for q)

$\vdash . p' \vee p$ (R.4)

$\vdash . p \vee p'$ (m.p.)

The law of the excluded middle says, "either p is true or p is false," or, "either p is true or not-p is true."

R.6 $\vdash : p \supset p' . \supset . p'$

Proof: $\vdash : p \vee p . \supset . p$ (R.Pp.1)

$\vdash : p' \vee p' . \supset . p'$ (sub. p' for p)

$\vdash : p \supset p' . \supset . p'$ (R.Df.1)

This is a special case of the principle of reductio ad absurdum. A more general form of this principle will be stated without proof as R.7.

R.7 $\vdash :p\supset q.\supset :p\supset q'.\supset .p'$

R.8 $\vdash .p\supset (p')'$

Proof: $\vdash .p\vee p'$ (R.5)

$\vdash .p'\vee (p')'$ (sub. p' for p)

$\vdash .p\supset (p')'$ (R.Df.1)

R.9 $\vdash .p\vee [(p')']'$

Proof: $\vdash :.q\supset r.\supset :p\vee q.\supset .p\vee r$ (R.Pp.5)

(1) $\vdash :.p'.\supset .[(p')']' .\supset :p\vee p'.\supset .p\vee [(p')']'$ (sub. p' for q and $[(p')']'$ for r)

$\vdash .p.\supset .(p')'$ (R.8)

(2) $\vdash .p'.\supset .[(p')']'$ (sub. p' for p)

$\vdash :p\vee p'.\supset .p\vee [(p')']'$ ((1), (2) and m.p.)

$\vdash :p\vee p'$ (R.5)

$\vdash .p\vee [(p')']'$ (m.p.)

R.10 $\vdash .(p')'\supset p$

Proof: $\vdash :p\vee q.\supset .q\vee p$ (R.Pp.3)

$\vdash :p\vee [(p')']'.\supset .[(p')']'\vee p$ (sub. $[(p')']'$ for q)

$\vdash .p\vee [(p')']'$ (R.9)

$\vdash .\ [(p')']'$ (m.p.)

$\vdash .\ (p')'\supset p$ (R.Df.1)

These are some of the basic relationships involved in multiple negations.

Three more important theorems will be stated without proof.

R.11 $\vdash .p'.\equiv .\ [(p')']'$

R.12 $\vdash .p.\equiv .\ (p')'$

This is the law of double negation.

R.13 $\vdash :p \supset q . \equiv . q' \supset p'$

This is the law of contraposition.

The complement of the principle of reductio ad absurdum is as follows:

R.14 $\vdash :p' \supset p . \supset . p$

Proof: $\vdash :. q \supset r . \supset : p \supset q . \supset . p \supset r$ (R.1)

$\vdash :. p \supset (p')' . \supset ; p' \supset p . \supset . p' \supset (p')'$ (sub. p for q, (p')' for r, and p' for p)

$\vdash . p \supset (p')'$ (R.8)

(1) $\vdash :p' \supset p . \supset . p' \supset (p')'$ (m.p.)

$\vdash :p \supset p' . \supset . p'$ (R.6)

(2) $\vdash :p' \supset (p')' . \supset . (p')'$ (sub. p' for p)

(3) $\vdash :p' \supset p . \supset . (p')'$ ((1), (2), Syll)

(4) $\vdash . (p')' \supset p$ (R.10)

$\vdash :p' \supset p . \supset . p$ ((3), (4), Syll)

R.15 $\vdash :p . \supset . p \vee q$

Proof: $\vdash :q . \supset . p \vee q$ (R.Pp.2)

$\vdash :p \vee q . \supset . q \vee p$ (R.Pp.3)

$\vdash :q . \supset . q \vee p$ (Syll)

$\vdash :p . \supset . p \vee q$ (sub. p for q and q for p)

The feature of material implication that a false proposition implies any proposition is shown by the following theorem.

R.16 $\vdash :p' . \supset . p \supset q$

Proof: $\vdash :p . \supset . p \vee q$ (R.15)

$\vdash :p' . \supset . p' \vee q$ (sub. p' for p)

$\vdash :p' . \supset . p \supset q$ (R.Df.1)

Thus, if a is a false statement, this implies "$\vdash . a'$." And, if b is any statement, substitution of "a" for "p" and "b" for "q" in R.16 gives "$\vdash : a' . \supset . a \supset b$"; then applying modus ponens gives "$\vdash . a \supset b$."

Another feature of material implication is that a true proposition is implied by any proposition. This is given by the following theorem.

R.17 $\vdash : q . \supset . p \supset q$

Proof: $\vdash : q . \supset . p \vee q$ (R.Pp.2)

$\vdash : q . \supset . p' \vee q$ (sub. p' for p)

$\vdash : q . \supset . p \supset q$ (R.Df.1)

A consequence of the definition of the logical product (R.Df.2) in P.M., is the equivalence of the law of the excluded middle and the law of contradiction. The law of contradiction is given as follows:

R.18 $\vdash . (p . p')'$

This states that it is false that a proposition is both true and false. If it should happen in a given system that both p and not-p could be asserted, then using R.16 it would follow that all propositions in the system would be provable, and the system would be inconsistent. This final point is of major importance when dealing with any formal system which seeks to be consistent. It so happens that this is a central problem to the second school of mathematical thought.

CHAPTER V

FORMALISM

The second major school in the foundations of mathematics is called formalism. It was founded around the turn of the century by David Hilbert. The method which this school employs is known as meta-mathematics. In the formalist thesis, the axiomatic development of mathematics is pushed to its extreme. Mathematics is considered to be a formal symbolic system whose abstract development consists of terms which are mere symbols and of theorems which are formulas involving these symbols. The foundations of mathematics do not lie in logic, but rather in the strings of symbols plus the rules of operations for obtaining new formulas from those previously developed. From this point of view, mathematics is devoid of concrete meaning and contains only abstract elements.

It so happens though, that the meta-language, i.e., the language which does not belong to the system, but is used to describe the system, does have meaning. Thus, from the basic rules expressed in the meta-language, can be developed meta-theorems which constitute the essence of meta-mathematics. The meta-theorems are statements about the marks in the system under study, plus statements about moves which can or cannot be made in that system according to its rules. Probably the most important question in meta-mathematics concerns the consistency of the system of meaningless symbols. Without a proof

of consistency, the whole study would be essentially senseless.

The success of the formalist program thus hinges on the solution of the consistency problem. With the crisis in classical mathematics from the paradoxes of set theory, it becomes even more important to find a consistency proof to save classical mathematics. Now the older consistency proofs, based upon models, merely shifted the question of consistency from one mathematical system to another. The formalists conceived a new direct approach to the consistency problem. This involved proving, by finite, constructible methods, that a contradictory formula can never occur in the system. This amounted to showing that if p is any theorem of the system, then not-p must not also occur as a provable theorem. If it can be shown that no such contradictory formulas are possible, then the system is consistent.

Hilbert developed a "proof theory" to deal with proofs of consistency. With this, he was able to give proofs for certain elementary systems; but in 1931, Kurt Gödel showed by methods acceptable to followers of any of the three main schools of mathematical philosophy, that it is impossible for a sufficiently complex formal system, such as the formalist's system for classical mathematics or _Principia Mathematica_, to prove consistency of the system by methods belonging to the system. Also, Gödel was able to demonstrate that for systems complex enough to deduce the natural numbers

and elementary arithmetic, consistency is incompatible with completeness. In other words, such systems can be consistent only at the price of being incomplete, and can be complete only at the price of being inconsistent. Thus, to work with a consistent system means that the system must fail to derive as theorems all the truths about natural numbers. This result is a decisive blow against the idea that mathematical truth can be identified with a deductive formal system. With these points as a background, the third school of mathematical thought called "intuitionism" will be examined.

CHAPTER VI

INTUITIONISM

During the first part of this century, the school of modern intuitionism was founded. The Dutch mathematician L. E. J. Brouwer is usually considered to be the founder. After writing his thesis in 1908 on the limitations of the law of the excluded middle as a mathematical tool, he then proceeded to espouse and develop a type of mathematics in line with this philosophy.

One of the fundamental convictions of the intuitionist school is that if mathematics is properly practiced and understood, it is a wholly autonomist and self-sufficient activity, i.e., it does not need the justification of logic or proofs of consistency which the logicists and formalists use for support.

The intuitionists contend that the antinomies which have originated in the foundations of mathematics are but a symptom that mathematics has not been properly pursued. They believe that logicism and formalism, in attempting to secure the foundations of classical mathematics, have used methods which are not truly mathematical. Thus, the intuitionists attempt to rebuild mathematics at all levels using only truly mathematical methods. "Mathematics" for the intuitionists is to perform mathematical constructions in the medium of pure intuition and to communicate the results to others so that they can be repeated.

Before considering the intuitionists mathematical program, the intuitionists mathematical philosophy should be examined briefly. Brouwer acknowledges his debt to Kant for providing certain basic tenents used in his philosophy of mathematics. Brouwer, as Kant, regards mathematical theories as synthetic, in the sense that propositions can be separated with a mutually exclusive and jointly exhaustive classification of being either analytic or synthetic.* Kant considered that the theorems and axioms of arithmetic and geometry are synthetic a priori, i.e., they validly describe space and time and constructions therein through pure intuition. Brouwer rejects Kant's intuition of space, but does accept his doctrine of the pure intuition of time and regards this to be the substratum of mathematics. Like Kant, Brouwer regards such intuition to be independent of any sense-perception. Thus, the subject-matter of intuitionist mathematics is intuited non-empirical objects and constructions which are self-evident through introspection.

Brouwer, in presenting his program of mathematical foundations, formulates two acts or insights of intuition. To quote Brouwer verbatim:

> The first act of intuitionism completely separates mathematics from mathematical language, in particular from the phenomena of language which are described by theoretical logic, and recognizes that intuitionist mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time, i.e., of the

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*For an explanation of these terms, see the appendix.

> falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the two-ity thus born is divested of all quality, there remains the empty form of the common substratum of all two-ities. It is this common substratum, this empty form, which is the basic intuition of mathematics.*

One point which Brouwer has brought out is that mathematics is essentially languageless. Just as the experience of, say, rowing a boat is not to be confused with its linguistic description and communication to others; in a similar manner the experience of mathematical intuitions and constructions must not be confused with its linguistic description and communication. Rowing a boat does not depend upon language; likewise, mathematical development, with its intuitive insights and constructions is languageless. According to Brouwer, the principles of classical logic are linguistic rules which are used in description and communication, but not in the activity itself of constructing proofs. Mathematics is essentially independent of both language and logic, which are but inessential aids.

With the distinction between the activities of mathematical constructions and the linguistic activity (including logic), an important question arises with regard to whether the linguistic representation does not outrun the construction. This follows from the fact that in everyday communication, language sometimes outruns its subject-matter, e.g., "unicorns." This danger is usually regarded as very small in the use of

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*L. E. J. Brouwer, "Historical Background, Principles and Methods of Intuitionism," South African Journal of Science, October, November, 1952, p. 142.

mathematical language. But, according to Brouwer, this does also happen in mathematics. In the case of mathematicians who employ the law of the excluded middle in dealing with infinite collections of mathematical objects, Brouwer believes that language is overrunning and misrepresenting the nature of mathematics. Thus, in certain cases, the law of the excluded middle must be rejected as an instrument for discovering new mathematical truths.

The second act of intuition for Brouwer concerns mathematics of the "potential infinite." This avoids the perceptually and intuitively empty notion of actual, pre-existing infinite totalities employed in classical mathematics. The notion of potential infinity consists of conceiving an indefinitely proceeding sequence, i.e., a sequence which can be generated step-by-step ad infinitum, but which is never complete. For the purpose of such a sequence, that of the natural numbers 1, 2, 3, 4, . . . is most basic and useful. The intuitionist asserts that, in this manner, new mathematical entities can be generated from those previously acquired. The terms of an infinitely proceeding sequence can be generated either by the free choice of the creating subject from mathematical entities previously acquired, or by means of a defining property which is supposable for entities previously acquired and which holds under the relation of equality between entities.

Intuitionist mathematics has several other points of difference with mathematics that is supported by a logical

substructure or expressed by a formalization. Its program is formulated simply enough, but it does involve some difficulties for the non-intuitionist because of its unfamiliar methods and procedure.

One of the most important points of the intuitionist program concerns the nature of mathematical existence. The logicist and formalist, as well as the classical mathematician, have allowed as legitimate pure existence-theorems which state that "there exists" a number with certain properties even though so far no method for constructing this number is known. This approach is utilized, for example, in Cantor's proof that there are more real numbers than there are natural numbers. Such proofs the intuitionist does not allow in his mathematics. For him, "mathematical existence" means the same as "actual constructibility," which is performed in a finite number of steps.

Another mathematician who has been instrumental in developing the intuitionist mathematics is Arend Heyting. His book _Intuitionism--An Introduction_, is considered to be the only comprehensive introduction to intuitionism written in English. It was Heyting, in 1930, who first formulated the logic of intuitionist mathematics. Thus, a consideration of his views should shed considerable light on the nature of intuitionist mathematics and logic.

The distinction between conceiving the natural numbers as an indefinitely proceeding sequence and that of classical

mathematics, which conceives them as forming a completed, infinite totality is illustrated by the following example from Heyting. Define two natural numbers c and d by these two rules: (A). c is the greatest prime such that c-1 is also prime; if no such prime exists, let c=o. (B). d is the largest prime such that d-2 is also prime; if no such prime exists, let d=o. Now from rule (A), it can be determined that c=3 since c-1 will be an even number. For (B), the classical concept of the natural numbers N as a completed totality would state that either there exists an infinite series of twin primes, in which case d=o, or there does not exist such a series and in which case d is the larger of the greatest pair of twin primes. Thus, (B) would also define some number d. The intuitionist, on the other hand, would not admit this argument since it does not give any definite information concerning the existence of an infinity of twin primes, and it would be impossible to generate natural numbers such that a constructible existence of d could be demonstrated. It would make no sense to the intuitionist in this case to use the law of the excluded middle and state that "either the sequence of twin primes is finite or it is not finite," unless it is backed by a constructive proof.

Now the classical mathematician may not be interested in any considerations outside of mathematics itself. Many tend to view a position such as that of the intuitionist as being "philosophical confusion," and not being concerned with

true mathematics. But, ironically enough, it is this very same criticism which the intuitionist applies to classical mathematics.

Concerning Heyting's example, the classical mathematician might object that the extent of our knowledge of the existence or non-existence of a last pair of twin primes is entirely irrelevant in questions of mathematical truth. Either an infinity of such primes exist, and d=o; or their number is finite and d equals the greatest prime such that d-2 is also prime. In every conceivable case d is defined, so it should not matter if d cannot actually be calculated.

The intuitionist rejects this argument since it is metaphysical in nature. If "to exist" does not mean "to be constructed," its meaning must lie in some "theory of reality." Of course, a mathematician may privately hold any metaphysical belief he likes, but it is not the task of the mathematician to investigate the meaning of such theories in mathematics. Brouwer's program, then, entails that mathematics is the study of something much simpler than metaphysics. And, in the study of mental mathematical constructions, "to exist" must be synonymous with "to be constructed."

An obvious objection to the intuitionist theory on the part of the classical mathematician might be to say that as long as it is unknown whether a last pair of twin primes exists, (B) is not a definition of an integer, but if this problem should be solved, it suddenly becomes a legitimate

definition. If, on June 1, 1980, it is proved that an infinity of twin primes exists; from that moment d=o. The classical mathematician might well ask if d=o before that date or not.

Heyting answers this question by again stating that, for the intuitionist, a mathematical assertion affirms the fact that a mathematical construction has been effected. Thus, it would follow that before the construction was made, it had not been made; and, therefore, before June 1, 1980, it had not been proved that d=o. But this is not the sense in which the question was asked. In order to clarify the classical mathematician's question, it seems that one must again refer to metaphysical concepts: to some world of mathematical objects existing independently of the mathematician, where d=o is true or false in some absolute sense. This, of course, relates to the question of whether mathematical "truths" are discovered or created. The intuitionist position is that such questions should be left to the mathematical philosopher, and such notions should not be allowed in mathematics. Heyting would admit that all mathematicians, including intuitionists, are convinced that in some sense mathematics bears upon eternal truths, but if such notions are allowed to enter mathematics itself, one gets involved in a maze of metaphysical difficulties. Thus, for the intuitionist, the study of mathematics is the study of mathematical constructions as such, and for this study, classical logic is inadequate. Also, Heyting states that the subject of constructive mathematical thought determines uniquely the

intuitionist premise and places it beside, not interior to, classical mathematics, which by its metaphysical considerations, studies another subject, whatever that subject may be.

The intuitionists also have many points of difference with the formalists. The formalists begin with the fact that in daily speech or non-formalized mathematical language, no word has a perfectly fixed meaning. Thus, the only way to achieve absolute rigour is to abstract all meaning from the mathematical statements and to consider them only as sequences of symbols, neglecting the sense they may convey. Then, by using a meta-mathematical formulation of definite rules for deducing new statements from those already known, the uncertainty from ambiguity of language can be avoided.

The intuitionist, on the other hand, is not interested in the formal side of mathematics; but he is interested in that type of reasoning which appears in meta-mathematics, since it involves the finite, constructible proofs which the intuitionist demands. This is developed to the farthest consequences, from the conviction that this type of reasoning results from one of the most fundamental faculties of the human mind.

It is true that a formalization of the finished part of intuitionist mathematics is possible. But this may only be considered as a linguistic expression, in a particularly suitable language, of mathematical thought. And, since any language

has a fundamental ambiguity, one cannot be mathematically sure that the formal system fully represents any domain of mathematical thought; this fact being verified by Gödel's incompleteness theorem.

A similar consideration applies to logic. The intuitionist does not consider logic to be the infallible criterion to verify mathematical thought. Even though this paper centers around intuitionist logic, logic is not the foundation for intuitionist mathematics. For it would follow that logic would need a foundation, which would involve principles more complex and less clear than those of mathematics itself. The intuitionist asserts that a mathematical construction should be so clear to the mind and its result so immediate that it needs no foundation whatsoever. All that is needed is what Heyting calls a "clear scientific conscience." By this one evaluates the constructions, which have as their starting point concepts clear even to young children.

To further clarify the point that intuitionist mathematics does not depend upon classical logic, it can be stated that whereas, for example, the logicist justifies his mathematics by an appeal to logic, the intuitionist justifies his logic by an appeal to mathematics. That is to say, the intuitionist builds his system without the use of any logic other than what he can justify by his mathematical constructions. Thus, the intuitionist is not concerned with logic in general, but rather with mathematical logic, i.e., a formulation of the

principles employed in making mathematical constructions. The intuitionist's view, then, is that formal systems of logic are essentially linguistic by-products of the languageless activity of mathematical construction and are mainly of pedagogical value.

From a purely formal point of view, intuitionist logic can be viewed as a subsystem of classical logic. Since any intuitionist mathematical proof would be acceptable in classical mathematics, but not all proofs of classical mathematics are acceptable to the intuitionist; it follows similarly that certain parts of classical logic are not acceptable to the intuitionist, while all theorems deduced from intuitionist logic are valid in the classical theory.

To clarify the above statements, the intuitionist logic as developed by Heyting will be examined.

CHAPTER VII

INTUITIONIST LOGIC

Heyting, in his development, first gives some basic distinctions and definitions which are used in the intuitionist propositional calculus. The first concerns the nature of propositions.

For P.M., a proposition was understood to be any statement in which it is meaningful to say that its content is either true or false, whether or not it is known which term actually applies. Examples of propositions are: "6 is an integer," "7 is an even integer," "the sequence 0123456789 occures in the decimal expansion of π." The first proposition is true, the second false, and even though the truth-value of the third is not known, it is still considered meaningful to say that it must be either true or false. Also, when a proposition is stated, regardless of its truth-value, it is meant to imply that it is true. Upon being proven, it is then considered to be an assertion.

Heyting, on the other hand, deals with the concept of a proposition in the following way: a mathematical proposition expresses a certain expectation. For example, a proposition such as "the constant of proportionality c in the relationship A=cbh is a rational number" expresses the expectation that two integers a and b can be found such that $c=\frac{a}{b}$. The word "proposition" can possibly be better expressed as the

intention which is linguistically expressed by the proposition. The intention, as in the example, refers to a construction thought to be possible. Thus, for Heyting, the concept of a proposition takes on a different meaning than that in P.M.

An assertion for Heyting is the affirmation of a proposition, i.e., the fulfillment of an intention expressed by the proposition. The assertion "c is rational" means that the integers a and b have been found such that $c=\frac{a}{b}$. An assertion is distinguished from its corresponding proposition by the assertion sign " $\vdash$" as used in P.M. The affirmation of a proposition is not itself a proposition, but rather the fulfillment of the intention expressed in the proposition. It expresses the fact that a proof has been constructed.

To develop the intuitionist logic, certain logical constants are introduced. It should be remembered that P.M. used two logical constants, namely v (disjunction) and $\sim$ (negation). The symbols $\supset$ (implication) and $\cdot$ (conjunction) were then defined in terms of disjunction and negation. For example, P.M. defined $p \supset q$ as $\sim p v q$.

Heyting initially introduces four symbols, $\neg$, $\supset$, $\wedge$, and v standing respectively for negation, implication, conjunction, and disjunction. He makes it clear though, that for intuitionist logic these symbols are independent of one another. Thus, for example, $p \supset q$ is not the same as $\neg p v q$.

Heyting states that a logical function is a process for constructing a proposition from previously given propositions. Thus, $p \wedge q$ can be asserted if, and only if, both p and q can be asserted. Also, $p \vee q$ can be asserted if at least one of the intentions p or q can be asserted. These two logical functions very nearly correspond to the functions of conjunction and disjunction in P.M., except for the intuitionist requirement of constructive proofs in asserting these propositions.

In considering the material implication $p \supset q$ of P.M., the truth-value of the implication depended upon the truth-values of the constituent propositions p and q. Thus, $p \supset q$ was false only when p was true and q was false.

Comparatively, the intuitionist implication is not a truth-function. For the intuitionist, $p \supset q$ can be asserted if, and only if, there has been performed some construction K which joined to a construction proving p would effect a construction proving q. Or, more concisely, a proof of p, together with K, would form a proof of q.

Concerning the concept of negation as a truth function, the intuitionists differ sharply with the logic of P.M. From P.M., $\sim p$ was the denial of a proposition p. Thus, $\sim p$ is true if p is false, and false if p is true.

For Heyting, negation is something thoroughly positive. It is the intention of a contradiction contained in the original intention. Hence, the proposition "B is not a prime" signifies the expectation that a contradiction will be derived

from the assumption that B is prime. It is important to note that the intuitionist negation must refer to a construction or proof procedure which leads to the contradiction. The symbol $\neg$ is used for negation. Thus, to affirm the proposition, $\vdash \neg p$, it is positively stated that a construction has been effected in one's mind which deduces a contradiction from the supposition that a proof of p has been effected. With this conception of negation then, the intuitionist theory diverges in certain aspects from that of the classical.

As a theorem in P.M., it is asserted that $\sim\sim p \supset p$. Using this, the logicist can assert p by showing that its negation is impossible and not necessarily demand a construction of p itself. To the intuitionist though, the impossibility of the impossibility of a property does not in all cases give a proof of the property itself. The following example will illustrate this point. Let the decimal expansion of π be written and under it the decimal expression $r=0.6666\ldots$ which will be terminated as soon as a sequence of digits 0123456789 has appeared in π. Now assume that r is not rational; then it follows that the sequence 0123456789 could not appear in π since this would imply that r is a terminated decimal expression, which is a rational number; but then $r=\frac{2}{3}$ which is also impossible. Thus, the assumption that r is not rational has led to a contradiction; yet it cannot be asserted that r is rational since this would mean that two integers a and b have been calculated such that $r=\frac{a}{b}$. This of course requires that

a sequence 0123456789 be found in π, or demonstrate that this cannot appear.

Also, the classical tautology $p \vee \neg p$ does not hold for the intuitionist unless p has been proved or reduced to a contradiction. To be more specific, the law of the excluded middle is questioned by the intuitionists only when the given proposition is not "intuitively" clear. This occurs when one tries to introduce actual infinite totalities into mathematics. For example, consider the proposition "there exists a prime number in set A." If it happens that A is a finite set, then the intuitionist would state that it is intuitively clear whether or not A contains the desired number since it can be determined in a finite number of steps. But, if A is an infinite set, the intuitionist would not allow the assertion of the proposition unless, either the prime number is exhibited in A, or it is shown that such an assumption leads to a contradiction. Thus, to state "either A contains a prime number or it does not" is mathematically meaningless unless a constructive proof can be determined which will verify it. Another more concrete example of the intuitionist's denial of the law of the excluded middle can be illustrated by "Goldbach's conjecture," which states that every even number can be expressed as the sum of two prime numbers. Despite much effort, mathematicians have not been able to prove nor disprove this proposition. In fact, there is no assurance that it will ever be solved. To the intuitionist's standard

of logical rigor, it is therefore meaningless to state that the conjecture is either true or false, unless one can provide a constructive proof for its verification or rejection.

As Heyting summarizes it, the formula $p \vee \neg p$ signifies the expectation of a mathematical construction. And, being a mathematical proposition, its validity is a mathematical problem which, when stated as a general law, is unsolvable by mathematical methods. It is in this sense that logic is dependent upon mathematics.

With the intuitionist's rejection of parts of classical logic, the next consideration should be to examine some of the points which the intuitionist logic does assert. For this, Heyting's propositional calculus will be developed.

<u>The Propositional Calculus</u>

For the reader who might be interested only in some of the results of the intuitionist logic, a few of the more important theorems will be listed, along with certain classical theorems which are not valid in the intuitionist theory. In the following statements, the assertion sign $\vdash$ will be placed in front of valid theorems, while the sign * will be placed in front of non-theorems.

(1) $\vdash .\ p \supset \neg\neg p$
$\quad\ ^{*}\neg\neg p \supset p$

The proposition p implies the double negation of p but not the converse.

(2) $\vdash. (p \supset q) \supset (\neg q \supset \neg p)$

$*(\neg q \supset \neg p) \supset (p \supset q)$

Only half of the law of contraposition holds.

(3) $\vdash. \neg p \supset \neg\neg\neg p$

$\vdash. \neg\neg\neg p \supset \neg p$

These two statements are thus equivalent.

(4) $*p \vee \neg p$

$\vdash. \neg\neg (p \vee \neg p)$

Even though the law of the excluded middle is not derivable in the system, the falsity of the falsity of this law is derivable.

(5) $\vdash. \neg (p \vee q) \supset \neg p \wedge \neg q$

$*\neg (p \wedge q) \supset \neg p \vee \neg q$

$\vdash. \neg (p \wedge \neg p)$

The last of these is of course the law of contradiction.

With this short consideration of some of the results of intuitionist logic, the formal development will now be examined. For this, the writer will be working with Heyting's original work, "Die formalen Regeln der intuitionistischen Logik." In this work, Heyting's proofs are little more than one or two line "directives"; and, for the most difficult proofs, he refers the reader to Peano's <u>Formulaire de mathematiques</u>. The writer will, for this thesis, develop all theorems which do not depend upon theorems whose proofs are from Peano. Thus, Heyting's own numbering will be employed

for the convenience of anyone who might want to refer to his work. With these points in mind, Heyting's logic will be developed.

I. Rules of Operation

1.1. The symbol $\vdash$ will be placed in front of a valid formula. If the formula is an axiom, the double symbol $\vdash\vdash$ will be used. (Heyting uses eleven axioms. They will be further emphasized by placing an "H" before them).

1.2. If a and b are asserted formulas, then $a \wedge b$ can be asserted. (This will be denoted as "conj")

1.3. If both a and $a \supset b$ can be asserted, then b can be asserted. (This will be denoted as "m.p.")

1.4. The statement "Const. a" at the beginning of a paragraph means that the symbol a is constant. Every symbol not introduced as a constant in this way is a variable. Whenever one replaces a variable throughout a formula by another symbol combination, then this in turn is also a valid formula.

1.5. $\binom{P}{X}a$ is the formula which results from formula a, when the variable X in a is replaced throughout by the symbol combination p. (Here, the symbol "sub" will be used instead.)

1.6. The formula $a\underline{D}b$ designates a definition: it means that to replace the symbol combination a by b (or b by a) in a valid formula, will produce a formula which is also valid.

II. Const. $\supset, \wedge, ., :, :., ::, \supset\subset, (,), \underline{D}, V, \neg.$

H.2.1. $\vdash \vdash .a \supset a \wedge a.$

H.2.11. $\vdash \vdash .a \wedge b \supset b \wedge a.$

H.2.12. $\vdash \vdash .a \supset b. \supset .a \wedge c \supset b \wedge c.$

H.2.13. $\vdash \vdash .a \supset b. \wedge .b \supset c. \supset .a \supset c.$

H.2.14. $\vdash \vdash .b \supset .a \supset b.$

H.2.15. $\vdash \vdash .a \wedge .a \supset b. \supset b.$

2.01. $\vdash .a \supset\subset b. \underline{D}. a \supset b. \wedge .b \supset a.$

2.2 $\vdash .a \wedge b \supset a.$

Proof: $\vdash .a \supset .b \supset a$ (2.14, sub b for a, a for b)

$\vdash .a \supset .b \supset a. \supset .a \wedge b \supset .b \supset a. \wedge b$ (2.12, sub b for c, $b \supset a$ for b)

(1) $\vdash .a \wedge b \supset .b \supset a. \wedge b$ (m.p.)

$\vdash .b \wedge .b \supset a. \supset a$ (2.15, sub b for a, a for b)

$\vdash .b \supset a. \wedge b. \supset .b \wedge .b \supset a$ (2.11, sub $b \supset a$ for a)

$\vdash .(b \supset a. \wedge b. \supset .b \wedge .b \supset a). \wedge .(b \wedge .b \supset a. \supset a)$ (conj.)

$\vdash .(b \supset a. \wedge b. \supset .b \wedge .b \supset a). \wedge .(b \wedge .b \supset a. \supset a). \supset (b \supset a. \wedge b. \supset a)$ (2.13, sub $(b \supset a. \wedge b)$ for a, $(b \wedge .b \supset a)$ for b, a for c)

(2) $\vdash .b \supset a. \wedge b. \supset a$ (m.p.)

$\vdash .(a \wedge b \supset .b \supset a. \wedge b). \wedge .(b \supset a. \wedge b. \supset a)$ ((1), (2), conj)

$\vdash .(a \wedge b \supset .b \supset a. \wedge b). \wedge .(b \supset a. \wedge b. \supset a). \supset . (a \wedge b. \supset a)$ (2.13, sub $a \wedge b$ for a, $(b \supset a. \wedge b)$ for b, b for c)

$\vdash .a \wedge b. \supset a$ (m.p.)

The writer would like to degress at this point to state that the rigor employed in proof 2.2 will be reduced slightly in the following way: consider the following example, replacing a, b, c by X, Y, Z respectively.

Example.	$\vdash . X \supset Y$	(given)
	$\vdash . Y \supset Z$	(given)
(1)	$\vdash . X \supset Y . \wedge . Y \supset Z$	(conj)
(2)	$\vdash . X \supset Y . \wedge . Y \supset Z . \supset . X \supset Z$	(2.13)
(3)	$\vdash . X \supset Z$	(m.p.)

In future proofs, steps (1), (2), and (3) will be combined into one step, (3), with the reason being stated as (conj, 2.13, m.p.). This may be used for relationships other than 2.13 also.

2.21 $\vdash . a \supset a$

Proof:	$\vdash . a \supset a \wedge a$	(2.1)
	$\vdash . a \wedge a \supset a$	(2.2, sub a for b)
	$\vdash . a \supset a$	(conj, 2.13, m.p.)

2.22. $\vdash . a \wedge b \supset b$

Proof:	$\vdash . a \wedge b \supset b \wedge a$	(2.11)
	$\vdash . b \wedge a \supset b$	(2.2, sub a for b, b for a)
	$\vdash . a \wedge b \supset b$	(conj, 2.13, m.p.)

The next theorem will be stated without proof.

2.27. $\vdash : a \supset . b \supset c : \supset\subset . a \wedge b \subset c$

Proof: (See Peano, <u>Formulaire de mathematiques</u>, vol. 1, section 1, para. 1, proofs 37 and 38)

2.02. $\vdash . a \wedge b \wedge c \ \underline{D} . a \wedge b . \wedge c$

H.3.1. $\vdash \vdash . a \supset a \vee b$

H.3.11. $\vdash \vdash . a v b \supset b v a$

H.3.12. $\vdash \vdash . a \supset c . \wedge . b \supset c . \supset . a v b \supset c$

3.2. $\vdash . a v b . v c . \supset . a v . b \vee c$

Proof:		$\vdash . a \supset a v b$	(3.1)
		$\vdash . a v b \supset b v a$	(3.11)
	(1)	$\vdash . a \supset b v a$	(conj, 2.13, m.p.)
		$\vdash . b v c . \supset a v . b v c$	(in (1) sub bvc for a, a for b)
		$\vdash . c \supset b v c$	(In (1) sub c for a)
	(2)	$\vdash . c \supset a v . b v c$	(conj, 2.13, m.p.)
	(3)	$\vdash . a \supset a v . b v c$	(3.1, sub bvc for b)
		$\vdash . b v c . \supset a v . b v c$	(in (1) sub bvc for a, a for b)
		$\vdash . b \supset b v c$	(3.1, sub b for a, c for b)
	(4)	$\vdash . b \supset a v . b v c$	(conj, 2.13, m.p.)
	(5)	$\vdash . a v b \supset a v . b v c$	((3), (4), conj, 3.12, m.p.)
		$\vdash . a v b . v c . \supset . a v . b v c$	((2), (5), conj, 3.12, m.p.)

3.01. $\vdash . a v b v c \underline{D} . a v b . v c$

3.22. $\vdash . a v a \supset a$

Proof:	$\vdash . a \supset a$	(2.1)
	$\vdash . a \supset a$	(2.1)
	$\vdash . a v a \supset a$	(conj, 3.12, m.p.)

H.4.1. $\vdash \vdash . \neg a \supset . a \supset b$

H.4.11. $\vdash \vdash . a \supset b . \wedge . a \supset \neg b . \supset \neg a$

4.2. $\vdash . a \supset b . \supset . \neg b \supset \neg a$

Proof:	$\vdash . \neg b \supset . a \supset \neg b$	(2.14, sub$\neg$b for b)

$\vdash . \neg b \supset . a \supset \neg b : \supset : \neg b . \wedge . a \supset b . \supset . a \supset \neg b . \wedge . a \supset b$
(2.12, sub $\neg b$ for a, $(a \supset \neg b)$ for b, $a \supset b$ for c)

$\vdash . \neg b . \wedge . a \supset b . \supset . a \supset \neg b . \wedge . a \supset b$ (m.p.)

$\vdash . a \supset b . \wedge . \neg b . \supset . \neg b . \wedge . a \supset b$ (2.11, sub $a \supset b$ for a, $\neg b$ for b)

$\vdash . a \supset b . \wedge . \neg b . \supset . a \supset \neg b . \wedge . a \supset b$ (conj, 2.13, m.p.)

$\vdash . a \supset \neg b . \wedge . a \supset b . \supset . a \supset b . \wedge . a \supset \neg b$
(2.11, sub $(a \supset \neg b)$ for a, $a \supset b$ for b)

$\vdash . a \supset b . \wedge . \neg b . \supset . a \supset b . \wedge . a \supset \neg b$ (conj, 2.13, m.p.)

$\vdash . a \supset b . \wedge . a \supset \neg b . \supset . \neg a$ (4.11)

$\vdash . a \supset b . \wedge . \neg b . \supset . \neg a$ (conj, 2.13, m.p.)

$\vdash . a \supset b . \wedge . \neg b . \supset . \neg a : \supset : a \supset b . \supset . \neg b \supset \neg a$
(2.27, sub $a \supset b$ for a, $\neg b$ for b, $\neg a$ for c)

$\vdash . a \supset b . \supset . \neg b \supset \neg a$ (m.p.)

4.21. $\vdash . a \supset \neg b . \supset . b \supset \neg a$

Proof: $\vdash . b \supset . a \supset b : \supset : b \wedge . a \supset \neg b . \supset . a \supset b . \wedge . a \supset \neg b$
(2.12, sub b for a, $a \supset b$ for b, $(a \supset \neg b)$ for c)

$\vdash . b \supset . a \supset b$ (2.14)

$\vdash . b \wedge . a \supset \neg b . \supset . a \supset b . \wedge . a \supset \neg b$
(m.p.)

$\vdash . a \supset \neg b . \wedge b . \supset . b \wedge . a \supset \neg b$
(2.11, sub $(a \supset \neg b)$ for a)

$\vdash . a \supset \neg b . \wedge b . \supset . a \supset b . \wedge . a \supset \neg b$
(conj, 2.13, m.p.)

$\vdash . a \supset b . \wedge . a \supset \neg b . \supset . \neg a$ (4.11)

$\vdash . a \supset \neg b . \wedge b . \supset . \neg a$ (conj, 2.13, m.p.)

$\vdash . a \supset \neg b . \wedge b . \supset . \neg a : \supset : a \supset \neg b . \supset . b \supset \neg a$
(2.27, sub $(a \supset \neg b)$ for a, $\neg a$ for c)

$\vdash . a \supset \neg b . \supset . b \supset \neg a$ (m.p.)

4.3. $\vdash . a \supset \neg\neg a$

Proof: $\vdash . \neg a \supset \neg a$ (2.21, sub $\neg a$ for a)

$\vdash . \neg a \supset \neg a . \supset . a \supset \neg\neg a$ (4.21, sub $\neg a$ for a, a for b)

$\vdash . a \supset \neg\neg a$ (m.p.)

4.31. $\vdash . \neg a \supset \neg\neg\neg a$

Proof: $\vdash . \neg a \supset \neg\neg\neg a$ (4.3, sub a for a)

4.32. $\vdash . \neg\neg\neg a \supset \neg a$

Proof: $\vdash . a \supset \neg\neg a$ (4.3)

$\vdash . a \supset \neg\neg a . \supset . \neg\neg\neg a \supset \neg a$ (4.2, sub $\neg\neg a$ for b)

$\vdash . \neg\neg\neg a \supset \neg a$ (m.p.)

4.4. $\vdash . a \wedge \neg a \supset b$

Proof: $\vdash . a \wedge \neg a \supset a$ (2.2, sub $\neg a$ for b)

$\vdash . a \wedge \neg a \supset \neg a$ (2.22, sub $\neg a$ for b)

$\vdash . a \wedge \neg a \supset a . \wedge . a \wedge \neg a \supset \neg a$ (conj)

$\vdash . a \wedge \neg a \supset a . \wedge . a \wedge \neg a \supset \neg a . \supset . \neg(a \wedge \neg a)$ (4.11, sub $(a \wedge \neg a)$ for a, a for b)

$\vdash . \neg(a \wedge \neg a)$ (m.p.)

$\vdash . \neg(a \wedge \neg a) \supset . a \wedge \neg a \supset b$ (4.1, sub $(a \wedge \neg a)$ for a)

$\vdash . a \wedge \neg a \supset b$ (m.p.)

4.41. $\vdash . a \wedge \neg a . \vee b . \supset b$

Proof: $\vdash . a \wedge \neg a \supset b$ (4.4)

$\vdash . b \supset b$ (2.21, sub b for a)

$\vdash . a \wedge \neg a . \vee b . \supset b$ (conj, 3.12, m.p.)

4.53. $\vdash . \neg a \vee \neg b \supset \neg(a \wedge b)$

Proof: $\vdash . a \wedge b \supset a$ (2.2)

$\vdash . a \wedge b \supset a . \supset . \neg a \supset \neg(a \wedge b)$
(4.2, sub $a \wedge b$ for a, a for b)

(1) $\vdash . \neg a \supset \neg(a \wedge b)$ (m.p.)

$\vdash . a \wedge b \supset b$ (2.22)

$\vdash . a \wedge b \supset b . \supset . \neg b \supset \neg(a \wedge b)$
(4.2, sub $a \wedge b$ for a)

(2) $\vdash . \neg b \supset \neg(a \wedge b)$ (m.p.)

$\vdash . \neg a \vee \neg b \supset \neg(a \wedge b)$ ((1), (2), conj, 3.12, m.p.)

4.8 $\vdash . \neg\neg(a \vee \neg a)$

Proof: $\vdash . a \supset a \vee \neg a$ (3.1, sub $\neg a$ for b)

$\vdash . a \supset a \vee \neg a . \supset . \neg(a \vee \neg a) \supset \neg a$
(4.2, sub $(a \vee \neg a)$ for b)

(1) $\vdash . \neg(a \vee \neg a) \supset \neg a$ (m.p.)

$\vdash . \neg a \supset \neg a \vee a$ (3.1, sub a for a, a for b)

$\vdash . \neg a \vee a \supset a \vee \neg a$ (3.11, sub a for a, a for b)

$\vdash . \neg a \supset a \vee \neg a$ (conj, 2.13, m.p.)

$\vdash . \neg a \supset a \vee \neg a . \supset . \neg(a \vee \neg a) \supset \neg\neg a$
(4.2, sub $\neg a$ for a, $(a \vee \neg a)$ for b)

(2) $\vdash . \neg(a \vee \neg a) \supset \neg\neg a$ (m.p.)

$\vdash . \neg\neg(a \vee \neg a)$ ((1), (2), conj, 4.11, m.p.)

This last theorem is, of course, the falsity of the falsity of the law of the excluded middle.

To illustrate the intuitionist logic which has been developed, Heyting gives the following example. Let A designate the

property of an integer being divisible by 27, B the same by 9, C the same by 3. For 27a can be written as 9x3a; by this mathematical construction K it follows that A entails B or ($A \supset B$). A similar construction J shows $B \supset C$. By effecting first K, then J (juxtaposition of K and J) it follows that $27a=3x(3x3a)$ showing $A \supset C$. This process remains valid if for A, B, C other arbitrary properties are substituted: If the construction K shows that $A \supset B$ and J shows that $B \supset C$, then the juxtaposition of K and J shows that $A \supset C$. Thus, there results a logical theorem. The process by which it is deduced does not differ essentially from mathematical theorems; it is only more general, e.g., in the same sense that "multiplication of integers is commutative" is a more general statement that "3x4=4x3." Thus, every logical theorem which has been developed is but a mathematical theorem of extreme generality.

Mathematical Foundations

At this point, the reader will be given a glimpse of the actual foundations of intuitionist mathematics. This will in no way be complete or rigorous, but it will give some idea of how intuitionist mathematics is conceived. Again, for the most part, this will be given in the way of examples which will illustrate the given ideas. It should be emphasized that it would be wrong to consider these examples as an essential part of intuitionist mathematics, just as it

would be wrong to think that the continuous non-differentiable function of Weierstrass is an essential part of classical integral calculus. The actual intuitionist mathematics is developed in a manner similar to classical mathematics except in a more restricted and rigorous manner.

The question as to the foundations or starting point of intuitionist mathematics is of prime importance. Heyting states that in the perception of an object, one is able to conceive the entity by abstracting from its particular qualities. Also, one has the ability to conceive of an indefinite repetition of entities. In these notions lies the source of the concept of natural numbers. This elementary notion of natural numbers is fundamental to intuitionistic mathematics. It is this notion which is assumed "intuitively." Heyting does not claim for it any form of absolute certainty, which he feels is unrealizable, but it is considered to be sufficiently clear to build mathematics upon. In fact, consistency is found to be a by-product of intuitionist mathematics.

The concept of a natural number, for Heyting, is suitable for three main reasons.

(1) It is easily understood by anyone with a minimum of education.

(2) It is universally applicable in the process of counting.

(3) It underlies the construction of analysis.

Instead of thinking in terms of axioms, Heyting thinks in terms

of evidence. Thus, axioms are not to be arbitrarily accepted or rejected. The natural numbers from the beginning possess certain properties which are detectable by simple examination. It so happens that the properties described by Peano's axioms are among them.

Except for mathematical induction, Peano's axioms are considered by the intuitionist to be intuitively-obvious properties embodied in the generation of the natural numbers. The first two properties, which state that 1 is a number and the immediate successor of a number is a number, can immediately be seen to be true by carrying out the generating construction. This same consideration applies to the third and fourth axioms, which state that 1 is not the successor of any number and that no two numbers have the same immediate successor. As for the induction property, it can be argued as follows: Let $Q(X)$ be a property of natural numbers such that $Q(1)$ holds, and $Q(n)$ implies $Q(n+1)$. Then given any natural number b, the intuitionist observes that by starting at 1 and passing over all natural numbers to be in a step-by-step generating process, if the property Q is preserved at each step, it can be therefore verified for b as well as its successor $b+1$ in a finite number of steps.

Analogous remarks apply for the recursive definitions of sum and product for natural numbers. By running over $1 \to p$, it follows that $a+p$ and $a \cdot p$ are defined for arbitrary natural numbers a and p. Thus, from the fundamental methods of induction and recursion, the arithmetic of natural numbers can be developed.

One concept needing clarification at this point is that of "equality." Heyting states that in reality a natural number must be fixed by means of a material representation, e.g., a mark on paper. For if a natural number is nothing but the result of a mental construction, it would be impossible to compare it with another natural number constructed at another time and place, since it would not subsist after the act of its creation. Thus, by a physical representation, one is able to compare by simple inspection natural numbers constructed at different times.

Another difference between intuitionism and classical mathematics appears when it comes to defining real numbers. In classical mathematics a real number can be defined by a Cauchy sequence of rational numbers. This can be defined as follows: The sequence of rational numbers $a_1, a_2, a_3, \ldots$ or, briefly, $\{a_n\}$ is a Cauchy sequence if for every natural number m there exists a natural number n, a function of m, such that for every natural number p, $|a_{n+p}-a_n| < \frac{1}{m}$.

The corresponding definition of an intuitionist Cauchy sequence can be formulated in almost the same way. The only difference consists in replacing the phrase "there exists" by the phrase "There can effectively be constructed" or "There can be effectively found." The following example will be used to illustrate.

Consider the following Cauchy sequences.

The first sequence $\{a_n\}$ is defined: $\frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \ldots$ or $\{\frac{2}{n}\}$. In this sequence each term can be effectively constructed, e.g., the hundredth term is $\frac{2}{100}$. Consider the second sequence $\{b_n\}$ defined as follows: If the n^{th} digit after the decimal point in the decimal expression of $\pi=3.1415\ldots$ is the 0 of the first sequence 0123456789 which occurs in this expansion of π, $b_n=1$: in every other case $b_n=\frac{2}{n}=a_n$.

Since the sequences $\{b_n\}$ and $\{a_n\}$ differ in at most one term, it follows that $\{b_n\}$ is a Cauchy sequence in the classical sense. But, for the intuitionist, since a construction is not known which will show whether or not the critical term occurs in $\{b_n\}$, i.e., whether a sequence 0123456789 occurs in π, thus it cannot be asserted that $\{b_n\}$ is a Cauchy sequence in the intuitionist sense. An intuitionist Cauchy sequence is also called a "real number generator", and like $\{a_n\}$ in the example, must be constructible.

With the correspondence of existence with that of actual constructibility of number-generators, Heyting is led to define two equality relations between real number generators, that being "identity" and "coincidence." Two number generators $\{a_n\}$ and $\{b_n\}$ are identical (symbolically $a\equiv b$) if for every n, $a_n=b_n$. They coincide ($a=b$) if for every natural number K, some integer $n=n(K)$ can be found such that for every natural number p, then $|a_{n+p}-b_{n+p}|<\frac{1}{K}$.

If the required $n=n(K)$ cannot be found for every K, this does not imply that a and b do not coincide. For an

intuitionist negation, just as an intuitionist affirmation, must result from a construction rather than the absence of one. Thus only if $a=b$ is contradictory can it be stated that a and b do not coincide, i.e., $a \neq b$. In other words, $a \neq b$ if, and only if, a construction can be effected which will contradict the supposition that $a=b$. From this, it should be clear also that to prove $a \neq b$ is contradictory does not imply that $a=b$.

A final point which should be considered concerns the more general question of solvability of mathematical problems. It happens that, for the intuitionist, solvability depends upon provability. Heyting explains this idea in the following way. As has been previously stated, a proof of a proposition is a mathematical construction. The intention of such a proof yields a proposition, say p. If the proposition "the proposition p is provable" is symbolized by "+p," then "+" is a logical function, viz. "provability." The assertions $\vdash +p$ and $\vdash p$ have the same logical meaning. For, if +p is proved, the intention of a proof of p has been satisfied, i.e., p has been proved; and if p is proved, then the provability of p is also proved. On the other hand, the propositions p and +p are not identical, i.e., the intention to prove p is not the same as the intention to prove the provability of p. As an illustration, in computing some number, say t, it might happen that a particular rational number, say A, is contained for an unusually long time within a small (epsilon) interval of a

Cauchy sequence, within which t is being more narrowly enclosed so that at some point it is finally suspected that t=A, i.e., A is expected to be found within the interval for any given epsilon of width. The suspicion is that t=A and the "provability" of this intention would be +(t=A). But such a suspicion is in no way a proof that it will happen. Thus the proposition +(t=A) contains more than the proposition (t=A).

If both of the propositions are negated, the result is the two different propositions "$\neg p$" and "$\neg +p$"; plus the assertions "$\vdash \neg p$" and "$\vdash \neg +p$," which are also different. "$\vdash \neg +p$" means that the assumption of a construction of +p is contradictory, i.e., its "provability" is impossible; while the simple expectation p itself may not lead to a contradiction. Relating to the example, assume that the contradictoriness can be asserted to the assumption that a proof of t=A can be constructed, i.e., ($\vdash \neg +p$). But at the same time, it may still seem to be possible by further computation that t=A, i.e., it may not be possible to reach a contradiction of the proposition itself. In fact, it is conceivable that it might be possible to prove that the proposition could never be contradictory, ($\vdash \neg \neg p$), and thus could be asserted both "$\vdash \neg +p$" and "$\vdash \neg \neg p$." In this case, t=A would be unsolvable. Thus, one is able to gain some insight into the general nature of essentially unsolvable problems.

The distinction between p and +p vanishes if a construction is intended for p itself, since the possibility of a

construction can be proved only by the actual construction itself. Thus, if the domain of consideration is limited to only those propositions which require a construction, a consideration of provability is not needed. This restriction can be imposed by regarding every proposition as having the intention of a construction for its proof added to it. It is in this sense that intuitionist logic must be understood.*

To a classical mathematician, parts of intuitionist mathematics may seem unnecessarily complicated and tedious. But this may be due mostly to unfamiliarity. It sometimes happens that the most seemingly lucid and common-sensical theories are inconsistent, and thus it may be with classical mathematics. Indeed no antinomies have as yet been discovered in intuitionist mathematics.

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*P. Benacerraf and H. Putnam, Philosophy of Mathematics, (selected readings) Prentice-Hall, Englewood Cliffs, N. J., 1964, pp. 48-49.

CHAPTER VIII

CONCLUSION

From the above development it can be seen that intuitionist mathematics contains very few arbitrary assumptions. And from this follows the fact that intuitionist mathematics is not plagued with inconsistencies. But, on the other hand, not everyone wants to concede the existence of some faculty of intuition. Also, the strongest objection to the intuitionist mathematics results from the fact that parts of classical mathematics are sacrificed because of the intuitionist demand for constructive proofs, as well as their rejection of axioms not felt to be intuitively obvious. Such is their rejection of Zermelo's axiom of choice which can be stated as follows: If a set A is divided into a collection of mutually disjoint nonempty subsets Q, R, S, . . . there exists at least one set X which has as its elements exactly one element from each of the subsets Q, R, S, . . .

Another way of expressing this axiom is to state that, for any set of nonempty sets, there always exists a selector-function which selects one member from each of these sets. It is obviously possible to exhibit a selector-function for a set consisting of a finite number of finite sets. But when it comes to selecting one member from each of an infinite number of sets, the exhibition of the selector-function, as a feature of perceivable or constructible objects, is not possible.

With the publication of papers by Zermelo concerning the axiom of choice, sharp differences of opinion were expressed by many eminent mathematicians. These differences were partially philosophical in nature, but were also related to the various fields of study. For example, researchers in topology apparently accept the axiom with no hesitation, for there seems little evidence that any significant part of topology can be derived without its use. In algebra, the situation is quite different. Though certain developments in algebra are quite awkward without the axiom of choice, so much can be accomplished without it that algebraists tend to proceed as far as possible avoiding its use. Much of analysis can be established without assuming the axiom of choice, but when one reaches measure theory and those portions of modern analysis which are founded in topological ideas, its evasion is nearly impossible.

The intuitionist objection to the axiom of choice lies in their conception of mathematical existence. The axiom asserts the existence of a certain set X, but does not state any way for finding X, or even that it is possible to find it; the assertion is merely that set X exists. The intuitionist, of course, denies the mathematical existence of a set if there is no way of ascertaining the members of the set.

From the intuitionist rejection of the axiom of choice, much of topology and analysis is lost. This situation may change in the future though, since new insights and different

approaches may be found which will lead to proofs of classical theorems previously thought unconstructible.

Finally, even though most mathematicians would not give intuitionist mathematics a privileged status, it should be remembered that it is no longer possible to deduce mathematics from logic in the manner of Frege and Russell; or to prove that classical mathematics is consistent by Hilbert's finite methods. It is still possible to practice intuitionist mathematics as originally conceived.

APPENDIX

APPENDIX

To give the unfamiliar reader an idea of the "analytic-synthetic" distinction employed, the following simplified definitions of the terms used by Kant are outlined:

A priori knowledge--Knowledge that does not need to be justified by experience or sense data.

Empirical (or a posteriori) knowledge--Knowledge that requires justification by experience or sense data.

Basic statements are considered in the following manner:

Analytic statements--Statements whose truth-value depends upon its logical form or definitions. All analytic statements are a priori.

Synthetic statements--Statements whose verification is non-analytic. They are of two types:

(1) Synthetic and empirical--Statements verified by sense data.

(2) Synthetic and a priori--Statements about the physical world which are not dependent upon sense data.

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