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Cross Ratio

Peyton Burlingame
Pittsburg State University

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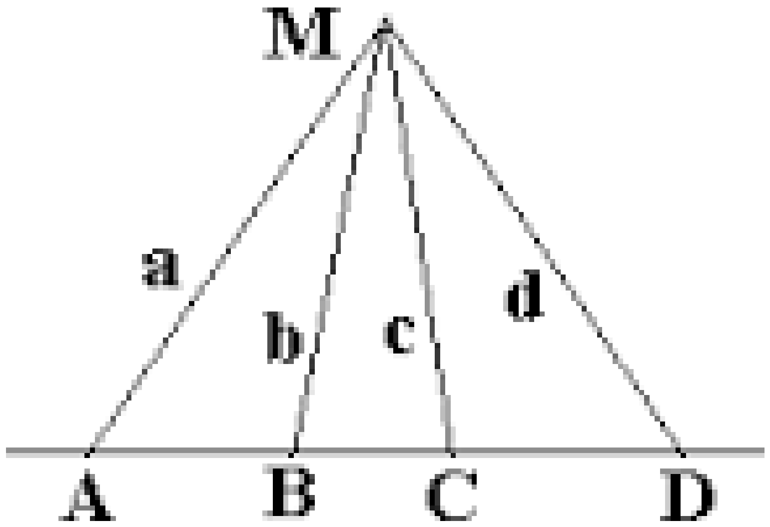
Problem

Given four points *A, B, C,* and *D* in order on a line in Euclidean space, under what conditions will there be a point *P* off the line such that the angles $\angle APB$, $\angle BPC$, and $\angle CPD$ have equal measure.

-MAA Monthly Problem #11915

Cross Ratio

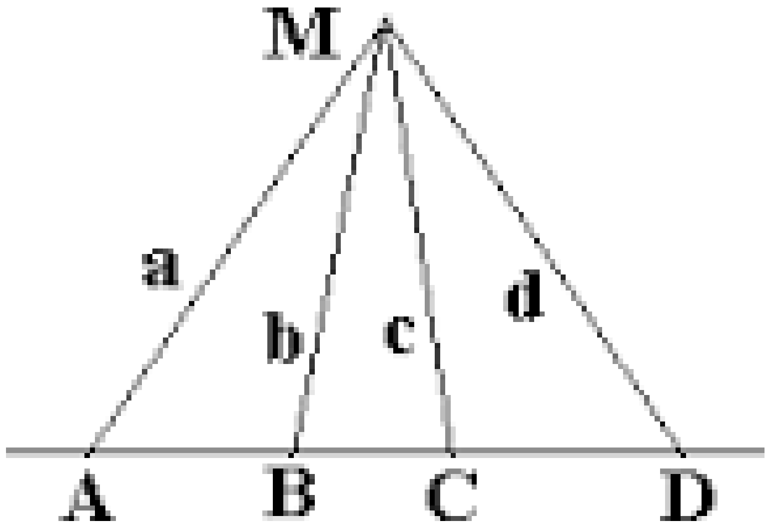
$$(ABCD) = \frac{AC}{CB} \Big/ \frac{AD}{DB}$$



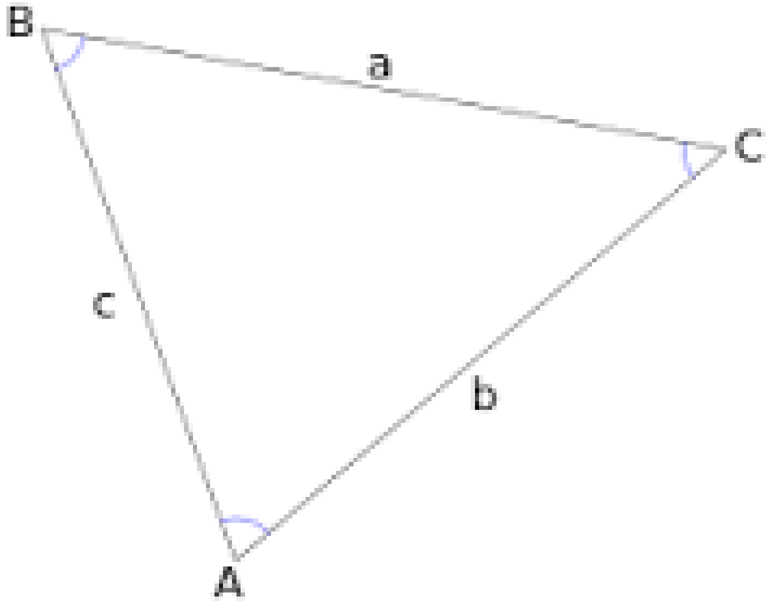
Geogebra Link

Cross Ratio Theorem

The cross ratio will be the same for any line crossing the four rays (*a, b, c, d)* starting at point *M.*



Law of Sines

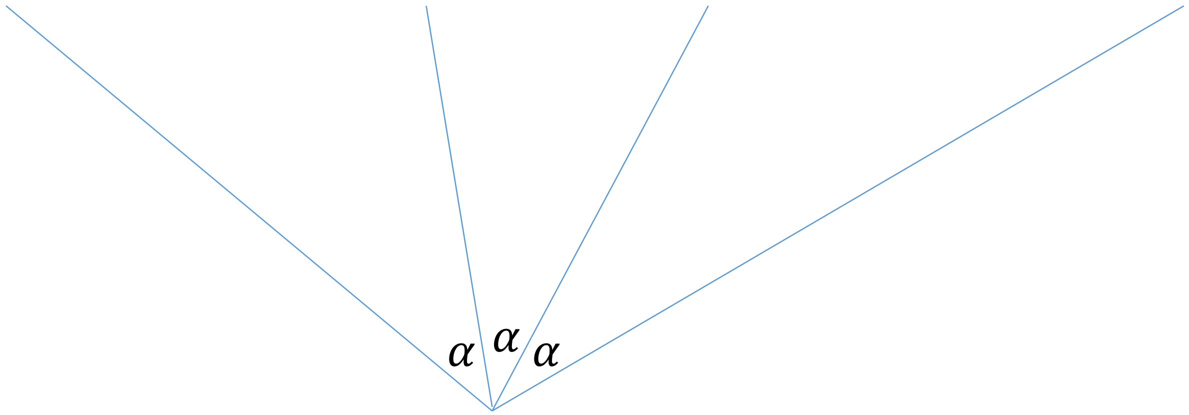


$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

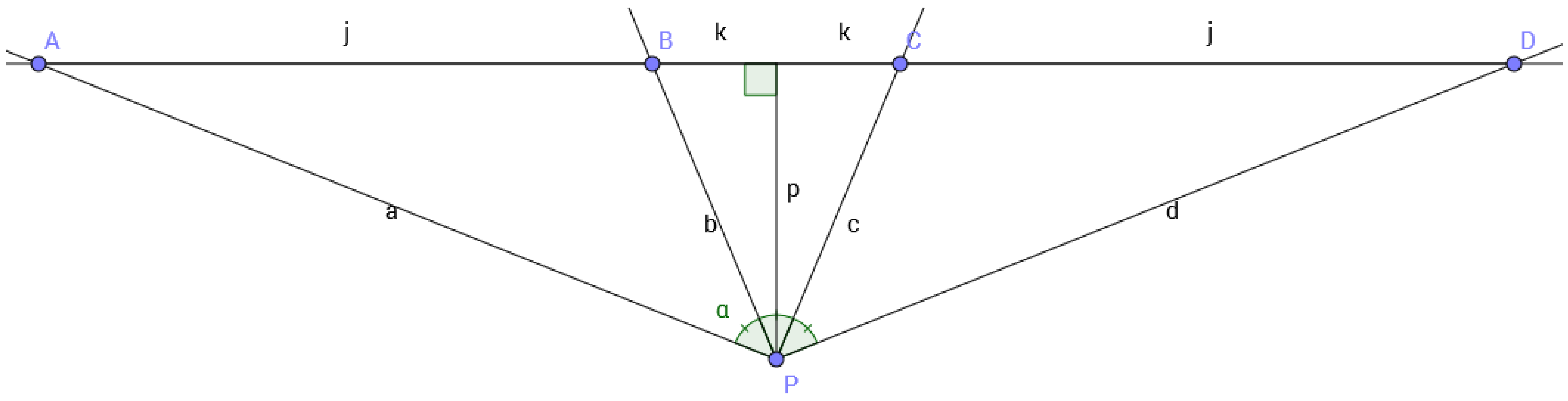
$$\Rightarrow \frac{\sin A}{\sin B} = \frac{a}{b}$$

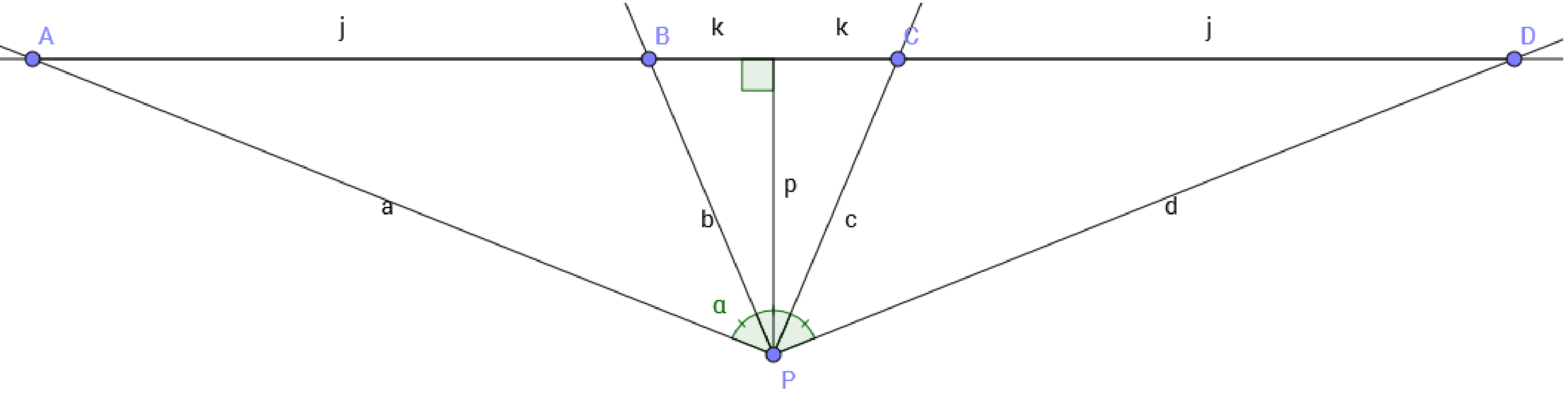
Cross Ratio Property

If the three angles are congruent, the cross ratio must be greater than 4/3.

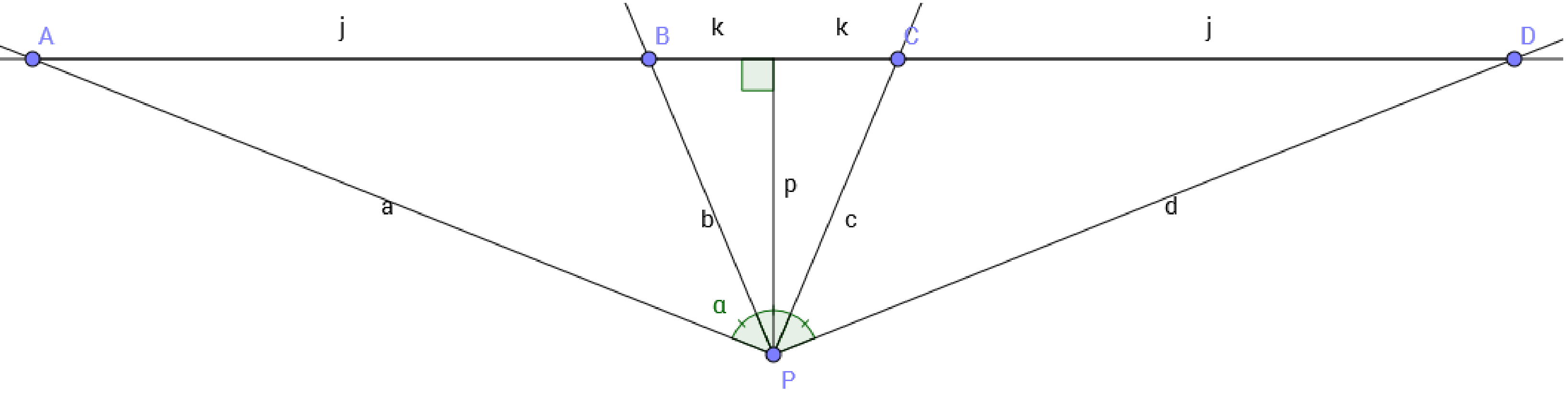


Best situation





$$(ABCD) = \frac{AC}{CB} \Big/ \frac{AD}{DB} = \frac{j+2k}{2k} \Big/ \frac{2j+2k}{j+2k} = \frac{(j+2k)^2}{2k(2j+2k)}$$



$$(ABCD) = \frac{(j+2k)^2}{2k(2j+2k)} = \frac{\left[\tan\left(\frac{3}{2}\alpha\right) + \tan\left(\frac{1}{2}\alpha\right)\right]^2}{4\tan\left(\frac{1}{2}\alpha\right) \times \tan\left(\frac{3}{2}\alpha\right)}$$

Cross Ratio is Increasing

On the interval $\left(0,\ \frac{\pi}{3}\right)$,

$$\frac{d}{d\alpha}\left(\frac{1}{2\cos(2\alpha)+1}+1\right) = \frac{4\sin(2\alpha)}{(2\cos(2\alpha)+1)^2} > 0$$

Cross Ratio Property

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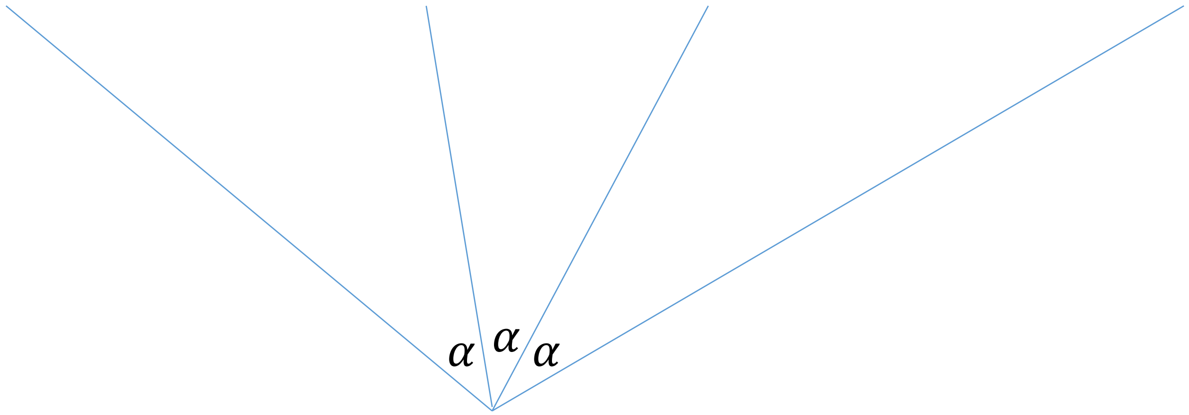

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Partial Solution

If there exists a *P* then the cross ratio is greater than $\frac{4}{3}$.

Areas of further work are necessary

Acknowledgments

- Dr. Leah Childers
- Appreciation to the Honors College for encouraging us to do this project

References

- Geogebra
- Wolfram Alpha
- Milne, John J. *An Elementary Treatise on Cross-ratio Geometry, with Historical Notes*. Cambridge: U, 1911. Print.