12-2020

OER Ellipses and traditional Rapanui houses on Easter Island

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A Multicultural Mathematical Activity

by Dr. Cynthia Huffman

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The Rapanui people of the island Rapa Nui, also known as Easter Island, at one time had a system of “writing” called Rongorongo. Unfortunately, the ability to read Rongorongo has since been lost. So currently archaeology and oral tradition are the only available sources for information about the early people of the island. Oral tradition states that the houses originally had upside down canoes for their roofs, and thus they are called *hare paenga*, or in English, boat-houses. In the literature, these boat houses are said to be elliptical in shape. In this activity, we will investigate the shape of archaeological remains of foundations of the boat houses to try to determine if they are indeed in the shape of an ellipse.

Replica hare paenga showing the stone foundation with holes for attaching the thatched roof.

Archaeological remains of a hare paenga (boat house) on Easter Island (photos by Dr. Cynthia Huffman, July 2019).

Review of Ellipses:

For simplicity, let us assume that our ellipse is set up on a coordinate system, centered at the origin, with the major axis in the vertical or y direction, and the minor axis in the horizontal or x

direction. One form for the equation of a generic ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $2a$ is the length of the minor axis and $2b$ is the length of the major axis.

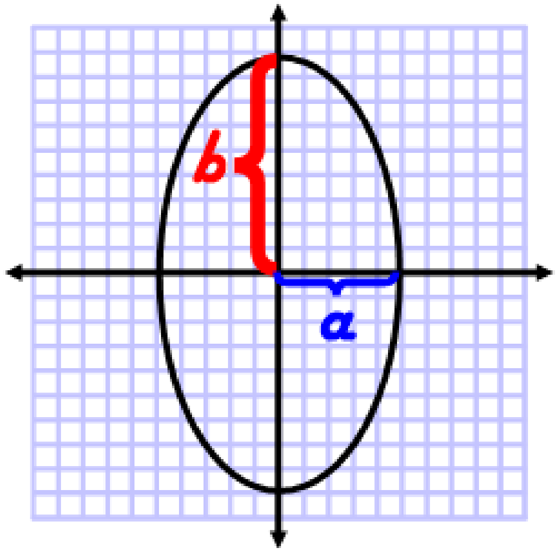
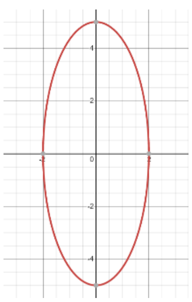
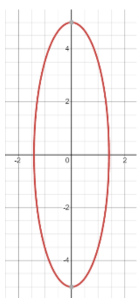
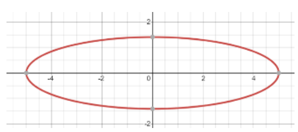
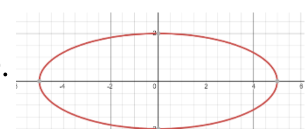


Figure 1

Exercise Set 1

1. Match the equation with the graph. (Graphs created using Desmos at desmos.com.)

A.  B.  C.  D. 

_____ $\frac{x^2}{25}+\frac{y^2}{4}=1$ _____ $\frac{x^2}{4}+\frac{y^2}{25}=1$ _____ $\frac{x^2}{25}+\frac{y^2}{2}=1$ _____ $\frac{x^2}{2}+\frac{y^2}{25}=1$

2. Write an equation of the following ellipses from the given information. Assume the major axis is vertical for each ellipse.

a. Major axis of 30 units and minor axis of 8 units

b. Major axis of 40 units and minor axis of 10 units

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3. For each ellipse give the length of the major axis and the minor axis, and circle whether the major axis is vertical or horizontal.

a. $\frac{x^2}{25}+\frac{y^2}{4}=1$

major axis = _____ units

minor axis = _____ units

vertical or horizontal

b. $\frac{x^2}{16}+\frac{y^2}{400}=1$

major axis = _____ units

minor axis = _____ units

vertical or horizontal

An ellipse is often defined as a planar shape consisting of the set of all points whose distances from two fixed points, called foci or focuses, add up to a constant sum. With an ellipse oriented as above on the previous page, with the major axis in the vertical direction, the foci would be located on the y-axis with each focus the same distance from the center of the ellipse. In the graph below, the distance from a focus to the origin is labelled c and the foci F_1 and F_2.

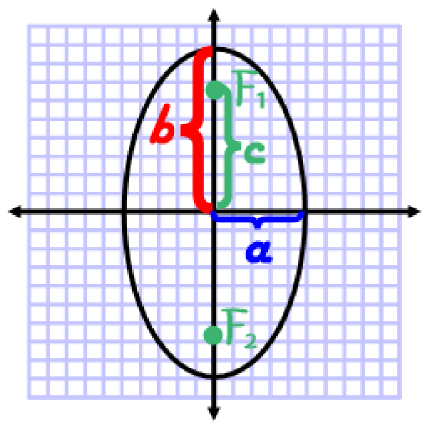

Figure 2

As mentioned earlier, if P is any point on the ellipse, then the sum of the distance from P to F_1 added to the distance P to F_2 is a constant.

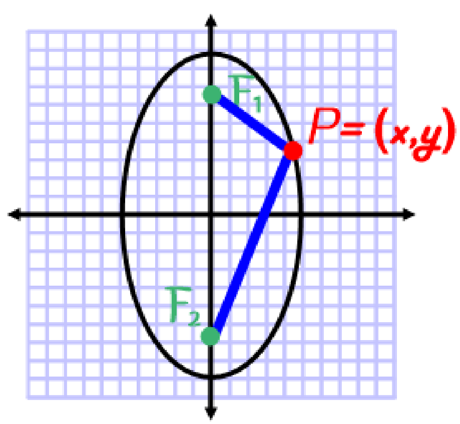

Figure 3

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We can determine this constant sum by considering a particular point, namely one of the endpoints of the major axis, say $(0,b)$.

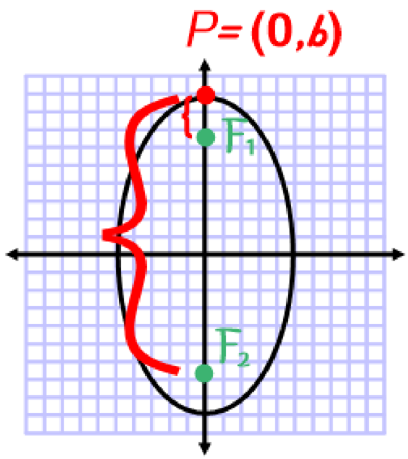

Figure 4

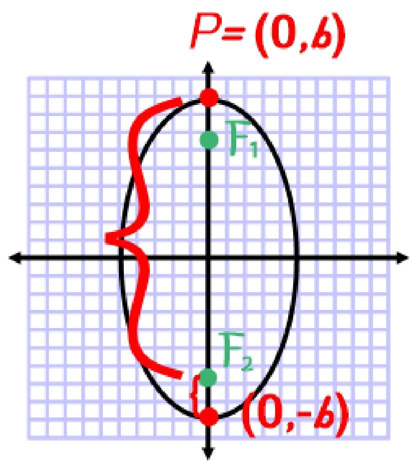

Figure 5

So the constant we are trying to determine is equal to the sum of the distance from $(0,b)$ to F_1 plus the distance from $(0,b)$ to F_2 (see Figure 4). Then, using the symmetry of the foci about the origin, the constant is also the sum of the distance from $(0,-b)$ to F_2 plus the distance from $(0,b)$ to F_2 (see Figure 5), which is the length of the major axis, $2b$ (see Figure 1).

Exercise Set 2

1. For each of the following ellipses, state the sum of the distances from any point on the ellipse to the foci.

a. $\frac{x^2}{16}+\frac{y^2}{400}=1$

sum is ___________

b. $\frac{x^2}{4}+\frac{y^2}{25}=1$

sum is ___________

2. Consider the ellipse $\frac{x^2}{25}+\frac{y^2}{625}=1$. If the distance from a point (x,y) on the ellipse to one of the foci is 13 units, what is the distance from that point to the other focus?

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Now, that we know the sum of the distances from any point on the ellipse to the foci is $2b$ (when the major axis is vertical), then it is also true for the particular point $(a,0)$ (see Figure 6). So, using the symmetry of the ellipse, we have that the distance from a focus to $(a,0)$ is b.

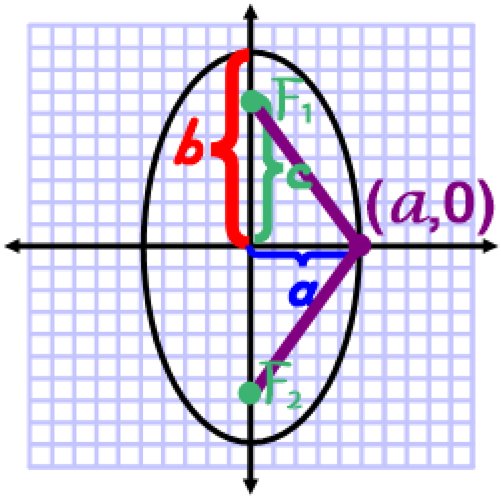

Figure 6

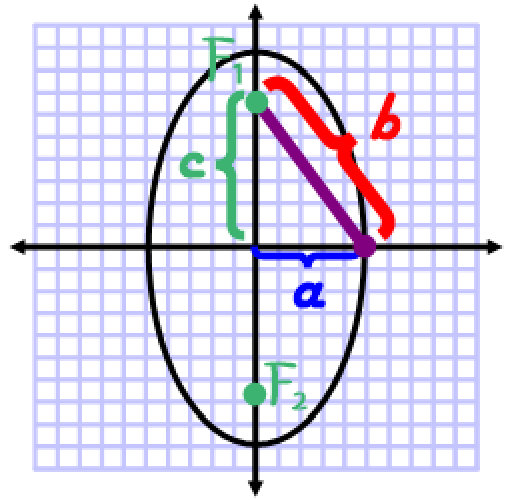

Figure 7

Consequently, by using the Pythagorean Theorem, we find that $c^2 + a^2 = b^2$ and $c = \sqrt{b^2 - a^2}$.

Thus, the general ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with major axis in the vertical direction, has major axis of length $2b$, minor axis of length $2a$, and the distance from a focus to the center of the ellipse is $c = \sqrt{b^2 - a^2}$. Therefore, the coordinates of the foci are $(0,c)$ and $(0,-c)$.

Exercise Set 3

1. Give the coordinates of the foci of each of the following ellipses.

a. $\frac{x^2}{16} + \frac{y^2}{25} = 1$ b. $\frac{x^2}{25} + \frac{y^2}{169} = 1$ c. $\frac{x^2}{18} + \frac{y^2}{50} = 1$

2. What is the formula for c, the distance from a focus to the center of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with major axis in the horizontal direction?

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Application of Ellipses to the Rapanui Boat-Houses (Hare Paenga):

If the shape of the Rapanui boat-houses were ellipses, then the foundation could have been laid out in the following way. Set two poles in the ground along the desired longer axis of the house, spaced evenly from the center of the house. Tie the ends of a rope to each pole so that the length of the rope when pulled taut is equal to the desired length of the house. Using a short pole or tool with a sharp end, pull the rope taut and trace out an ellipse on the ground.

To algebraically describe a boat-house foundation, one could measure across the widest spots in each of the two perpendicular directions of the foundation, and then substitute these lengths for a and b using the form of the equation of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. For example, if the length of a foundation was 26 feet and the width 10 feet, then the equation would be $\frac{x^2}{5^2}+\frac{y^2}{13^2}=1$ or $\frac{x^2}{25}+\frac{y^2}{169}=1$. If we let c be the distance from the center of the ellipse (in our case the origin) to a focus of the ellipse, then $c=\sqrt{b^2-a^2}$. Since we are letting b be the major axis, b^2-a^2 will be a positive value. For our example, $c^2=b^2-a^2=13^2-5^2=169-25=144$. So, c is 12 feet. Thus, the foundation could have been laid out using poles that were set $2c=24$ feet apart, using a rope that, after being tied to the two poles, had length $2b=26$ feet. So, the rope when pulled taut along the axis of the two poles, would have extended 1 foot past the nearest pole.

Exercise Set 4

1. Find the equation of the ellipse, centered at the origin and vertical major axis, with the following major and minor axes. (The measurements are taken from actual Rapanui archaeological ruins of hare paenga.)

a. major axis = 30 feet

minor axis = 5 feet

b. major axis = 46 feet

minor axis = 6 feet

2. For each of the ellipses in problem 1 above, determine the coordinates (to 2 decimal places) of the foci, where the poles would be placed.

a.

b.

3. For each of the ellipses in problem 1 above, determine how far (to the nearest inch) the rope would extend past each pole.

a. ______ inches

b. _____ inches

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Solutions to Exercises

Exercise Set 1

1. D, A, C, B
2. a. $b = 15$ and $a = 4$ so $\dfrac{x^2}{16} + \dfrac{y^2}{225} = 1$, b. $b = 20$ and $a = 5$ so $\dfrac{x^2}{25} + \dfrac{y^2}{400} = 1$
3. a. major axis = 5 units, minor axis = 2 units, horizontal

   b. major axis = 20 units, minor axis = 4 units, vertical

Exercise Set 2

1. a. Since $b = 20$, the sum is $2b = 40$. b. Since $b = 5$, the sum is $2b = 10$.
2. Since $b = 25$, the sum is $2b = 50$. So, the remaining length is $50 - 13 = 37$ units.

Exercise Set 3

1. a. $c = \sqrt{25-16} = \sqrt{9} = 3$, so the foci are at $(0,3)$ and $(0,-3)$.

   b. $c = \sqrt{169-25} = \sqrt{144} = 12$, so the foci are at $(0,12)$ and $(0,-12)$.

   c. $c = \sqrt{50-18} = \sqrt{32} = 4\sqrt{2}$, so the foci are at $\left(0, 4\sqrt{2}\right)$ and $\left(0, -4\sqrt{2}\right)$.

2. $c = \sqrt{a^2 - b^2}$

Exercise Set 4

1. a. $\dfrac{x^2}{25} + \dfrac{y^2}{900} = 1$, b. $\dfrac{x^2}{36} + \dfrac{y^2}{2116} = 1$
2. a. $c = \sqrt{900-25} = \sqrt{875} \approx 29.85$, so the foci are at $(0, 29.58)$ and $(0, -29.58)$.

   b. $c = \sqrt{2116-36} = \sqrt{2080} \approx 45.61$, so the foci are at $(0, 45.61)$ and $(0, -45.61)$.

3. a. $b - c \approx 30 - 29.58 = 0.42$ feet ≈ 5 inches

   b. $b - c \approx 46 - 45.61 = 0.39$ feet ≈ 5 inches

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