7-2020

A Real World Example of Solving a Quadratic Equation in Movie CGI

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by
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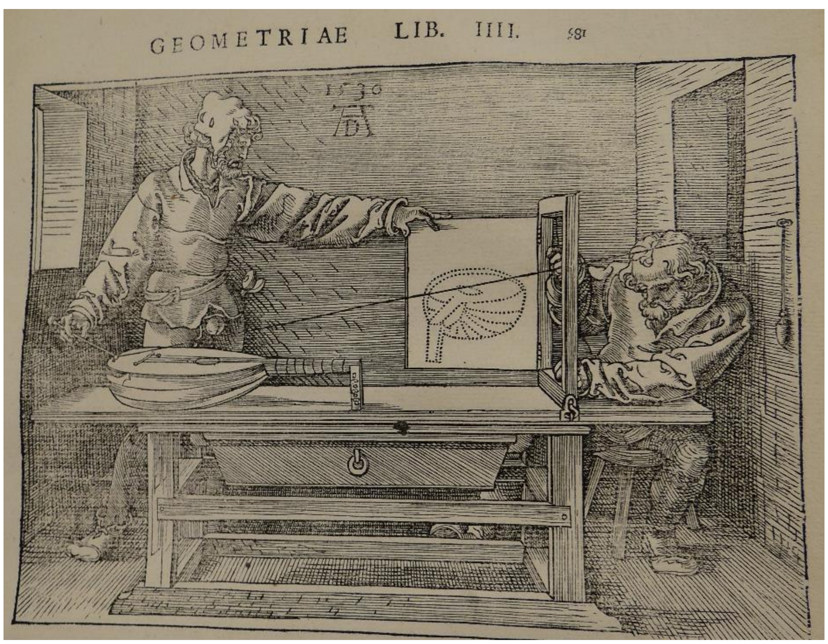
Introduction

Many students are motivated by real world applications of mathematics, and most students are familiar with the term CGI (Computer Generated Imagery) from watching movies and playing video games. This particular activity ties solving a quadratic equation to CGI. It is appropriate for use in a beginning algebra course in secondary school or in a college algebra class.

One of the techniques used in CGI is ray tracing. The concept of ray tracing is not new. In fact, it can be traced back to the 16th century. The artist and mathematician Albrecht Dürer (1471-1528) created a wood carving which shows a device using ray tracing and line of sight as an aid in perspective drawing.



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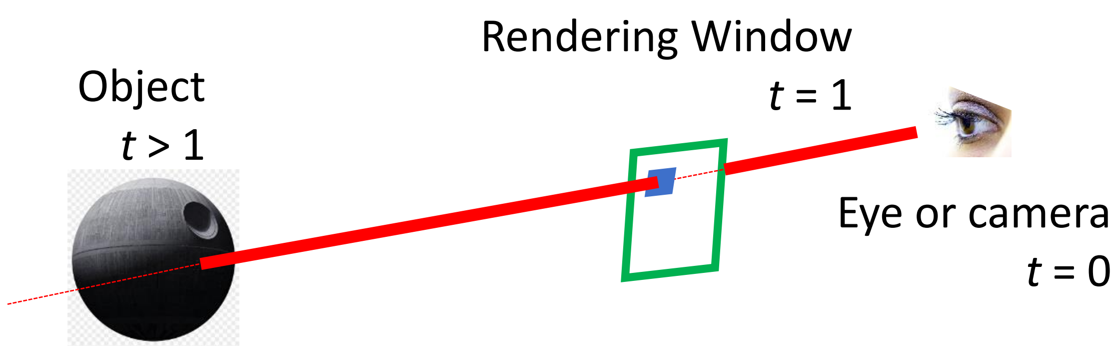
The idea of ray tracing is to use lines of sight which pass through a rendering window to hit objects in the scene. If the line hits a particular object, say a spherical ball, then the corresponding pixel in the rendering window is colored to match the object at the point of intersection of the line of sight with the object. Since we are working in 3 dimensions, the line is given parametrically as 3 linear equations in the parameter *t*. So, when a value is plugged in for *t*, a specific point (x, y, z) is given on the line, and if *t* ranges over all real numbers, the result is the entire line. (Note: The activity does NOT require that students are familiar with parametric equations.) Usually the parametric equations for the line are set up so that the eye or camera corresponds to *t* = 0 and the rendering window corresponds to *t* = 1, as in the picture below.



Example

Take for the line of sight, the line defined parametrically by $\begin{cases} x = 3t - 1 \\ y = t - 5 \\ z = 2t + 4 \end{cases}$, where *t* is any real number. Then for our CGI set-up, the camera is placed on the line at the point where *t* = 0. So, the camera is at the point $\begin{cases} x = 3(0) - 1 = -1 \\ y = 0 - 5 = -5 \\ z = 2(0) + 4 = 4 \end{cases}$ or $(-1, -5, 4)$. The rendering window is on the line at the point where *t* = 1, that is $\begin{cases} x = 3(1) - 1 = 2 \\ y = 1 - 5 = -4 \\ z = 2(1) + 4 = 6 \end{cases}$ or $(2, -4, 6)$. (In practice, these two points would be used to come up with the parametric equations for the line of sight by using an object known as a vector, rather than starting with the parametric equations.) Suppose a spherical

object (for example, the Death Star in Star Wars) is given by the equation $(x-7)^2+(y+1)^2+(z-10)^2=12$, and we would like to determine if the spherical object is in the line of sight. Mathematically, this corresponds to determining if the line of sight intersects the object, or in other words, finding out if there is a value for the parameter *t* in the line of sight which gives a point that is both on the line and on the sphere. So, we will substitute the expressions in the parameter *t* for *x*, *y*, and *z* in the equation of the sphere, as below.

$$\begin{aligned}
&(x-7)^2+(y+1)^2+(z-10)^2=12\\
&\left((3t-1)-7\right)^2+\left((t-5)+1\right)^2+\left((2t+4)-10\right)^2=12\\
&(3t-8)^2+(t-4)^2+(2t-6)^2=12\\
&9t^2-48t+64+t^2-8t+16+4t^2-24t+36=12\\
&14t^2-80t+104=0\\
&2(t-2)(7t-26)=0\\
&t=2,\ \frac{26}{7}\approx 3.7
\end{aligned}$$

So, there are two values of the parameter *t* where the line intersects the sphere, corresponding to the line first hitting the sphere at $t=2$, then passing through its interior, before leaving the sphere at $t=\frac{26}{7}\approx 3.7$. Thus, the line of sight enters the sphere at

$$\begin{cases} x=3(2)-1=5\\ y=2-5=-3\\ z=2(2)+4=8\end{cases} \text{ or } (5,-3,8) \text{ and exits the sphere at } \begin{cases} x=3\left(\frac{26}{7}\right)-1=\frac{71}{7}\\ y=\frac{26}{7}-5=-\frac{9}{7}\\ z=2\left(\frac{26}{7}\right)+4=\frac{80}{7}\end{cases} \text{ or } \left(\frac{71}{7},-\frac{9}{7},\frac{80}{7}\right).$$

Now that we've seen an example, complete the following similar problems. The solution is given below each problem. A handout for student use can be found at the end of this document.

Practice Problems with Solutions

Problem 1. Suppose the line of sight is given parametrically as $\begin{cases} x=3t-10\\ y=-2t+8\\ z=-t+3\end{cases}$.

a. If the camera is placed at the point $t=0$, then the coordinates of the point for the camera are (_____, _____, _____).

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b. If the rendering window is placed at the point $t=1$, then the coordinates of the point for the rendering window are (_____, _____, _____).

c. Suppose a spherical object is given by the equation $x^2+y^2+z^2=5$. Find the point(s), if any, of intersection of the line of sight with this sphere. (Hint: First find the value(s) of the parameter *t* for the intersection(s).)

d. Did the line of sight hit the sphere? In how many points? Does this make sense? Explain.

Solution 1: a. $(-10,8,3)$ b. $(-7,6,2)$ c. The quadratic to be solved is $14t^2-98t+168=0$, which can be solved by factoring or by using the quadratic formula, to obtain $t=3,4$. The intersection points are $(-1,2,0)$ when $t=3$, $(-2,0,-1)$ when $t=4$

d. yes, in 2 points, the line of sight passes through the interior of the sphere, first entering the sphere when $t=3$ and then exiting the sphere $t=4$

Problem 2. Suppose the line of sight is given parametrically as $\begin{cases} x=-3t+8 \\ y=-6t+15 \\ z=-t+5 \end{cases}$.

a. If the camera is placed at the point $t=0$, then the coordinates of the point for the camera are (_____, _____, _____).

b. If the rendering window is placed at the point $t=1$, then the coordinates of the point for the rendering window are (_____, _____, _____).

c. Suppose a spherical object is given by the equation $x^2+y^2+z^2=14$. Find the point(s), if any, of intersection of the line of sight with this sphere. (Hint: First find the value(s) of the parameter *t* for the intersection(s).)

d. Did the line of sight hit the sphere? In how many points? Does this make sense? Explain.

Solution 2: a. $(8,15,5)$ b. $(5,9,4)$ c. The quadratic to be solved is $46t^2-238t+300=2(23t-50)(t-3)=0$, which can be solved by factoring or by using the quadratic formula, to obtain $t=\frac{50}{23},3$. The intersection points are $\left(\frac{34}{23},\frac{45}{23},\frac{65}{23}\right)$ when $t=\frac{50}{23}\approx 2.17$, $(-1,-3,2)$ when $t=3$ d. yes, in 2 points, the line of sight passes through the interior of the sphere, first entering the sphere when $t=\frac{50}{23}$ and then exiting the sphere when $t=3$

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Problem 3. Suppose the line of sight is given parametrically as $\begin{cases} x = 4t - 6 \\ y = 3t + 4 \\ z = 3 \end{cases}$.

a. If the camera is placed at the point $t = 0$, then the coordinates of the point for the camera are (_____, _____, _____).

b. If the rendering window is placed at the point $t = 1$, then the coordinates of the point for the rendering window are (_____, _____, _____).

c. Suppose a spherical object is given by the equation $(x+1)^2 + (y-6)^2 + (z-3)^2 = 25$. Find the point(s), if any, of intersection of the line of sight with this sphere. (Hint: First find the value(s) of the parameter *t* for the intersection(s).)

d. Did the line of sight hit the sphere? In how many points? Does this make sense? Explain.

Solution 3: a. $(-6, 4, 3)$ b. $(-2, 7, 3)$ c. The quadratic to be solved is $25t^2 - 52t + 4 = (25t - 2)(t - 2) = 0$, which can be solved by factoring or by using the quadratic formula, to obtain the intersection points $\left(\frac{-142}{25}, \frac{106}{25}, 3\right)$ when $t = \frac{2}{25} = 0.08$ and $(2, 10, 3)$ when $t = 2$ d. yes, in 2 points, the line of sight passes through the interior of the sphere, first entering the sphere when $t = \frac{2}{25}$ and then exiting the sphere $t = 2$

Problem 4. Suppose the line of sight is given parametrically as $\begin{cases} x = t - 20 \\ y = 3t + 1 \\ z = t - 5 \end{cases}$.

a. If the camera is placed at the point $t = 0$, then the coordinates of the point for the camera are (_____, _____, _____).

b. If the rendering window is placed at the point $t = 1$, then the coordinates of the point for the rendering window are (_____, _____, _____).

c. Suppose a spherical object is given by the equation $x^2+y^2+z^2=382$. Find the point(s), if any, of intersection of the line of sight with this sphere. (Hint: First find the value(s) of the parameter *t* for the intersection(s).)

d. Did the line of sight hit the sphere? In how many points? Does this make sense? Explain.

Solution 4: a. $(-20,1,-5)$ b. $(-19,4,-4)$ c. The quadratic to be solved is $11t^2-44t+44=11(t-2)^2=0$, which can be solved by factoring or by using the quadratic formula, to obtain $(-18,7,-3)$ when $t=2$ d. yes, in 1 point, the line of sight hits the boundary of the sphere in just 1 point (the line is tangent to the sphere) when $t=2$

Problem 5. Suppose the line of sight is given parametrically as $\begin{cases} x=t+2 \\ y=2t+1 \\ z=3t-4 \end{cases}$.

a. If the camera is placed at the point $t=0$, then the coordinates of the point for the camera are (______, ______, ______).

b. If the rendering window is placed at the point $t=1$, then the coordinates of the point for the rendering window are (______, ______, ______).

c. Suppose a spherical object is given by the equation $(x-1)^2+(y-2)^2+(z+5)^2=1$. Find the value(s) of the parameter *t* for the intersection(s). What was different about your answer from the previous 4 problems?

d. Did the line of sight hit the sphere? Does this make sense? Explain.

Solution 5: a. $(2,1,-4)$ b. $(3,3,-1)$ c. The quadratic to be solved is $14t^2+4t+2=0$, which can be solved by using the quadratic formula, to obtain $t=\dfrac{-1\pm\sqrt{-6}}{7}$ or $\dfrac{-1}{7}\pm i\dfrac{\sqrt{6}}{7}$, no real solutions d. no, since there are no real solutions the

line of sight does not hit the sphere; in other words, the sphere is outside of this particular line of sight

Problem 6. a. Think about what was in common for all of the 5 previous problems. What do you notice?

b. In each case, to find the intersection(s) of a line of sight (given parametrically in *t*) with a sphere, we had to solve a quadratic equation in the variable *t*. How many real solutions can a quadratic equation have?

c. In how many points can a line intersect a sphere?

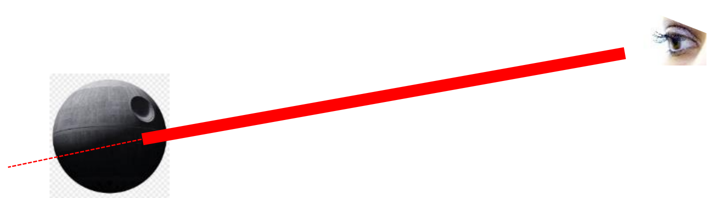
d. Match the geometrical possibilities on the left with the algebraic possibilities on the right.

Line of sight can intersect a sphere in	# of possible solutions of a quadratic equation
0 points	2 real roots
1 point	0 real roots (2 complex nonreal roots)
2 points	1 double real root

e. Draw pictures of the three possibilities of a line of sight and a spherical object, like the picture at the beginning of this handout.

Solution 6: a. In all cases, one had to solve a quadratic equation in order to find the points of intersection. b. A quadratic solution can have 0, 1, or 2 real roots. c. A line can pass through a sphere, intersecting it in 2 points or be tangent to the sphere, intersecting it in 1 point, or miss the sphere, intersecting it in 0 points. d. 0 points of intersection corresponds to 0 real roots; 1 point of intersection corresponds to 1 double real root of the quadratic equation; 2 points corresponds to 2 real roots of the quadratic equation

e. 2 points of intersection



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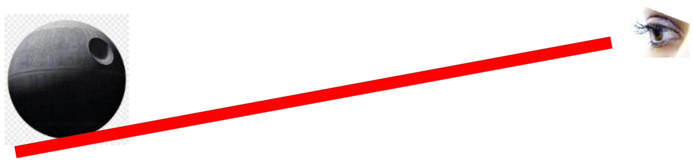
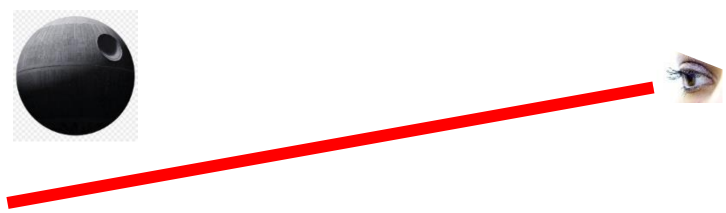
1 point of intersection



0 points of intersection



Obtaining a General Formula for a Line of Sight Hitting a Sphere

At some time, you may have played a video game in which you aimed a line at colored balls or a linear laser beam at spheres. Can you imagine how slowly the game would move if we had to solve for the intersections individually from scratch each time like in the practice problems above? Since we are solving the same type of problem each time, namely finding in three dimensions the intersection of a line with a sphere, this could be accomplished more quickly by plugging values into a general formula.

Suppose the line is given parametrically as $\begin{cases} x = a_1 t + b_1 \\ y = a_1 t + b_1 \\ z = a_1 t + b_1 \end{cases}$, and for simplicity, assume the sphere is centered at the origin with equation $x^2 + y^2 + z^2 = r^2$, where *r* is the radius of the sphere.

Challenge Problem

As in the practice problems and the preceding example, substitute the expressions for *x*, *y*, and *z* into the equation of the sphere, and simplify so that you have a quadratic equation of the form $At^2 + Bt + C = 0$, with

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$$A = \underline{\qquad\qquad\qquad\qquad}$$

$$B = \underline{\qquad\qquad\qquad\qquad}$$

$$C = \underline{\qquad\qquad\qquad\qquad}.$$

Then $t = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$. Check your answer with the values from Practice Problem 1.

For more of a challenge, complete the above problem using the general equation of the sphere with radius *r* and center (h, j, k), namely $(x-h)^2 + (y-j)^2 + (z-k)^2 = r^2$.

Solution to Challenge Problem

$$A = a_1^{\,2} + a_2^{\,2} + a_3^{\,2}$$

$$B = 2\left(a_1 b_1 + a_2 b_2 + a_3 b_3\right)$$

$$C = b_1^{\,2} + b_2^{\,2} + b_3^{\,2} - r^2$$

In particular with Practice Problem 1, $A = (3)^2 + (-2)^2 + (-1)^2 = 9 + 4 + 1 = 14$,
$B = 2\left(3(-10) + (-2)(8) + (-1)(3)\right) = 2(-30 - 16 - 3) = 2(-49) = -98$,
$C = (-10)^2 + (8)^2 + (3)^2 - 5 = 100 + 64 + 9 - 5 = 168$

For the extra challenge, $A = a_1^{\,2} + a_2^{\,2} + a_3^{\,2}$

$$B = 2\left(a_1 b_1 + a_2 b_2 + a_3 b_3 - a_1 h - a_2 j - a_3 k\right)$$

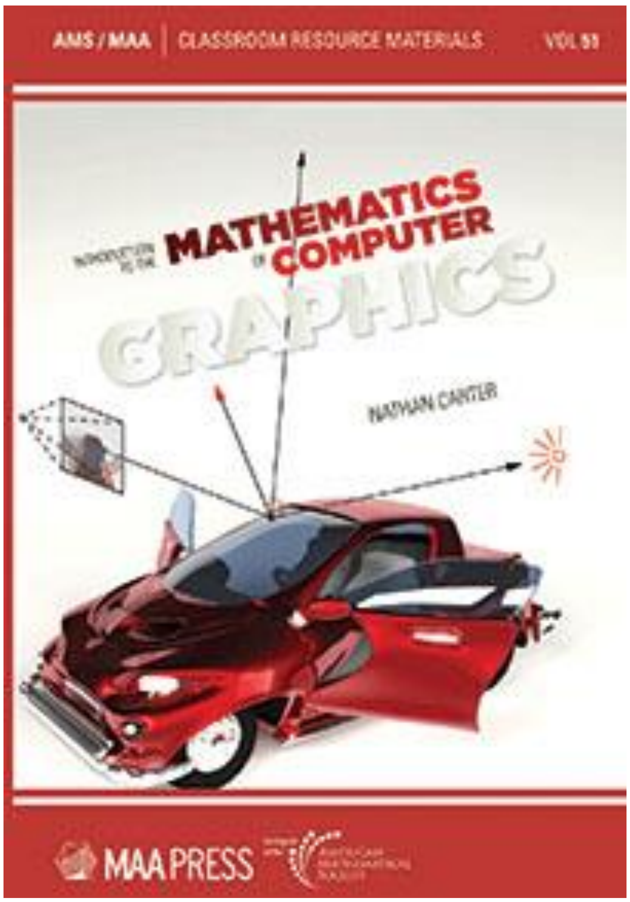
$$C = (b_1 - h)^2 + (b_2 - j)^2 + (b_3 - k)^2 - r^2$$

Conclusion

It is important to expose students to the beauty and usefulness of mathematics. Since computer graphics are familiar to most students due to video games and movies, they make a great source for motivating topics in mathematics. This activity shows students an application of solving quadratic equations to computing the line of sight to spherical objects in computer graphics.

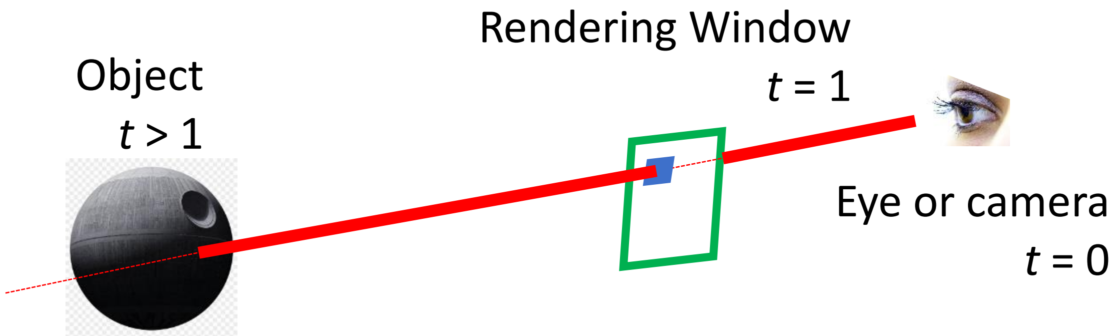
The author also has written an activity on matrix multiplication related to computer graphics that is suitable for a high school algebra or college algebra class. It can be found in the Pittsburg State University Digital Commons at https://digitalcommons.pittstate.edu/oer-math/16/ .

For more connections between mathematics and computer graphics, check out the book which provided the motivation for this activity - *Introduction to the Mathematics of Computer Graphics* by Nathan Carter, published in the Classroom Resource Materials series of MAA Press, an imprint of the American Mathematical Society, https://bookstore.ams.org/clrm-51/.



Name ________________________________

CGI and Solving Quadratics Handout



Example

Take for the line of sight, the line defined parametrically by $\begin{cases} x = 3t - 1 \\ y = t - 5 \\ z = 2t + 4 \end{cases}$, where t is any real number. Then for our CGI set-up, the camera is placed on the line at the point where $t = 0$. So, the camera is at the point $\begin{cases} x = 3(0) - 1 = -1 \\ y = 0 - 5 = -5 \\ z = 2(0) + 4 = 4 \end{cases}$ or $(-1, -5, 4)$. The rendering window is on the line at the point where $t = 1$, that is $\begin{cases} x = 3(1) - 1 = 2 \\ y = 1 - 5 = -4 \\ z = 2(1) + 4 = 6 \end{cases}$ or $(2, -4, 6)$. (In practice, these two points would be used to come up with the parametric equations for the line of sight by using an object known as a vector, rather than starting with the parametric equations.) Suppose a spherical object (for example, the Death Star in Star Wars) is given by the equation $(x-7)^2 + (y+1)^2 + (z-10)^2 = 12$, and we would like to determine if the spherical object is in the line of sight. Mathematically, this corresponds to determining if the line of sight intersects the object, or in other words, finding out if there is a value for the parameter t in the line of sight which gives a point that is both on the line and on the sphere. So, we will substitute the expressions in the parameter t for x, y, and z in the equation of the sphere, as below.

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$$(x-7)^2+(y+1)^2+(z-10)^2=12$$
$$\left((3t-1)-7\right)^2+\left((t-5)+1\right)^2+\left((2t+4)-10\right)^2=12$$
$$(3t-8)^2+(t-4)^2+(2t-6)^2=12$$
$$9t^2-48t+64+t^2-8t+16+4t^2-24t+36=12$$
$$14t^2-80t+104=0$$
$$2(t-2)(7t-26)=0$$
$$t=2,\ \frac{26}{7}\approx 3.7$$

So, there are two values of the parameter t where the line intersects the sphere, corresponding to the line first hitting the sphere at $t=2$, then passing through its interior, before leaving the sphere at $t=\frac{26}{7}\approx 3.7$. Thus, the line of sight enters the sphere at

$\begin{cases} x=3(2)-1=5 \\ y=2-5=-3 \\ z=2(2)+4=8 \end{cases}$ or $(5,-3,8)$ and exits the sphere at $\begin{cases} x=3\left(\frac{26}{7}\right)-1=\frac{71}{7} \\ y=\frac{26}{7}-5=-\frac{9}{7} \\ z=2\left(\frac{26}{7}\right)+4=\frac{80}{7} \end{cases}$ or $\left(\frac{71}{7},-\frac{9}{7},\frac{80}{7}\right)$.

Now that we've seen an example, complete the following similar problems.

1. Suppose the line of sight is given parametrically as $\begin{cases} x=3t-10 \\ y=-2t+8 \\ z=-t+3 \end{cases}$.

a. If the camera is placed at the point $t=0$, then the coordinates of the point for the camera are (_____, _____, _____).

b. If the rendering window is placed at the point $t=1$, then the coordinates of the point for the rendering window are (_____, _____, _____).

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c. Suppose a spherical object is given by the equation $x^2 + y^2 + z^2 = 5$. Find the point(s), if any, of intersection of the line of sight with this sphere. (Hint: First find the value(s) of the parameter *t* for the intersection(s) like in the example above.)

d. Did the line of sight hit the sphere? In how many points? Does this make sense? Explain.

2. Suppose the line of sight is given parametrically as $\begin{cases} x = -3t + 8 \\ y = -6t + 15 \\ z = -t + 5 \end{cases}$.

a. If the camera is placed at the point $t = 0$, then the coordinates of the point for the camera are (_____, _____, _____).

b. If the rendering window is placed at the point $t = 1$, then the coordinates of the point for the rendering window are (_____, _____, _____).

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c. Suppose a spherical object is given by the equation $x^2 + y^2 + z^2 = 14$. Find the point(s), if any, of intersection of the line of sight with this sphere. (Hint: First find the value(s) of the parameter *t* for the intersection(s).)

d. Did the line of sight hit the sphere? In how many points? Does this make sense? Explain.

3. Suppose the line of sight is given parametrically as $\begin{cases} x = 4t - 6 \\ y = 3t + 4 \\ z = 3 \end{cases}$.

a. If the camera is placed at the point $t = 0$, then the coordinates of the point for the camera are (_____, _____, _____).

b. If the rendering window is placed at the point $t = 1$, then the coordinates of the point for the rendering window are (_____, _____, _____).

c. Suppose a spherical object is given by the equation $(x+1)^2 + (y-6)^2 + (z-3)^2 = 25$. Find the point(s), if any, of intersection of the line of sight with this sphere. (Hint: First find the value(s) of the parameter *t* for the intersection(s).)

d. Did the line of sight hit the sphere? In how many points? Does this make sense? Explain.

4. Suppose the line of sight is given parametrically as $\begin{cases} x = t - 20 \\ y = 3t + 1 \\ z = t - 5 \end{cases}$.

a. If the camera is placed at the point $t = 0$, then the coordinates of the point for the camera are (_____, _____, _____).

b. If the rendering window is placed at the point $t = 1$, then the coordinates of the point for the rendering window are (_____, _____, _____).

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c. Suppose a spherical object is given by the equation $x^2 + y^2 + z^2 = 382$. Find the point(s), if any, of intersection of the line of sight with this sphere. (Hint: First find the value(s) of the parameter t for the intersection(s).)

d. Did the line of sight hit the sphere? In how many points? Does this make sense? Explain.

5. Suppose the line of sight is given parametrically as $\begin{cases} x = t + 2 \\ y = 2t + 1 \\ z = 3t - 4 \end{cases}$.

a. If the camera is placed at the point $t = 0$, then the coordinates of the point for the camera are (_____, _____, _____).

b. If the rendering window is placed at the point $t = 1$, then the coordinates of the point for the rendering window are (_____, _____, _____).

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c. Suppose a spherical object is given by the equation $(x-1)^2+(y-2)^2+(z+5)^2=1$. Find the value(s) of the parameter t for the intersection(s). What was different about your answer from the previous 4 problems?

d. Did the line of sight hit the sphere? Does this make sense? Explain.

6. a. Think about what was in common for all of the 5 previous problems. What do you notice?

b. In each case, to find the intersection(s) of a line of sight (given parametrically in t) with a sphere, we had to solve a quadratic equation in the variable t. How many solutions can a quadratic equation have?

c. In how many points can a line intersect a sphere?

d. Match the geometrical possibilities on the left with the algebraic possibilities on the right.

Line of sight can intersect a sphere in	**# of possible solutions of a quadratic equation**
0 points	2 real roots
1 point	0 real roots (2 complex nonreal roots)
2 points	1 double real root

e. Draw pictures of the three possibilities of a line of sight and a spherical object, like the picture at the beginning of this handout.

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Obtaining a General Formula for a Line of Sight Hitting a Sphere

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Challenge Problem

As in the practice problems and the preceding example, substitute the expressions for *x*, *y*, and *z* into the equation of the sphere, and simplify so that you have a quadratic equation of the form $At^2 + Bt + C = 0$, with

$A =$ ______________________

$B =$ ______________________

$C =$ ______________________.

Then $t = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$. Check your answer with the values from Problem 1.

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